Method of iterated kernels in problem of wave propagation in heterogeneous media: calculation of higher orders terms

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***Abstract*—The approximated solution of wave propagation problem in smooth heterogeneous media by use of the iterated kernels method is proposed. It represents the result of iterated method application to the integral equation equivalent to the Helmholtz scalar equation. The resulting solution has a compact type and unites the advantages of the Born scattering and short-wave asymptotic methods. The way of increasing accuracy of the solution on the basis of addition of terms is shown. Their functional form is determined by the requirement of meeting the conditions of the Helmholtz equation solution and represents a compromise between the accuracy and the simplicity of the solution.**

***Keywords*—wave propagation, method of iterated kernels.**

I. Introduction

Most of interesting to study media, are heterogeneous. The problem of wave propagation in a medium with arbitrary spatial dependence has no exact solutions and approximate methods have to be used. Existing approaches in describing a field that has passed through a heterogeneous medium can be divided into two main groups. These are methods that take into account multipath propagation (the Born method, for example) and methods with multiple interactions (the geometric optics method, the Rytov method, the parabolic equation method etc.) [1, 2]. These approaches have different applications: the first type efficiently describes the scattering of waves by concentrated objects, the second type describes the change in the radiation characteristics when a wave passes through a smoothly heterogeneous medium. In the series of articles [3-6], an attempt was made to create a method that combines the advantages of these approaches. The main idea consists of applying the method of iterated kernels to the integral equation

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2 \int\limits_V G_0(R_0) E(\mathbf{r}) \delta\varepsilon(\mathbf{r}) d\mathbf{r}, \qquad (1)$$

that describes the problem of wave propagation in an infinite heterogeneous medium in the scalar approximation. Here the $E_0(\mathbf{r}_0)$ represents the primary wave electric field strength, $\delta\varepsilon(\mathbf{r}) = (\varepsilon(\mathbf{r}) - \varepsilon_0)/\varepsilon_0$ is the disturbance of the medium's dielectric permittivity in relation to the background value ε_0, $G_0(R_0) = \dfrac{e^{ikR_0}}{4\pi R_0}$ is the Green's function of an infinite homogeneous medium with the wave number k, $R_0 = |\mathbf{r}_0 - \mathbf{r}|$. The unbounded space volume integral in the right part represents the field scattered by heterogeneities.

Applying the concept of resolvent [7], the solution for the integral equation (1) can be given by

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2 \int\limits_V E_0(\mathbf{r}) \Gamma(\mathbf{r}, \mathbf{r}_0) d\mathbf{r},$$

where $\Gamma(\mathbf{r}, \mathbf{r}_0)$ denotes a resolvent, which is determined by the Neumann series

$$\Gamma(\mathbf{r}, \mathbf{r}_0) = \sum_{n=0}^{\infty} k^{2n} W_{n+1}(\mathbf{r}, \mathbf{r}_0),$$

which converges in case of sufficiently small values of the wave number k. *n+1*-st iterated kernel $W_{n+1}(\mathbf{r}, \mathbf{r}_0)$ can be determined by the following recurrent relation [3]

$$W_{n+1}(\mathbf{r}, \mathbf{r}_0) = \int W_n(\mathbf{r}, \mathbf{r}') W(\mathbf{r}', \mathbf{r}_0) d\mathbf{r}',$$

$$W_1(\mathbf{r}, \mathbf{r_0}) = W(\mathbf{r}, \mathbf{r_0}) = G_0(R_0) \delta\varepsilon(\mathbf{r}).$$

The main difficulty of such approach lies in the awkwardness of iterated kernels writing represented by multidimensional integrals of rather complicated form that can't be summarized completely. Thus it is necessary either to be limited to a small quantity of considered kernels (the Born approximation, the double scattering theory [2], etc.) or to use simplifying approximations.

The most natural approach consist to Decomposition of an arbitrary dependence $\delta\varepsilon(\mathbf{r})$ into a complete system of basic functions and the subsequent exact calculation of the resulting integrals seems to be the most natural approach. Such method of calculation using the Taylor series was proposed in [3].

The solution in the linear approximation given by

$$\delta\varepsilon(\mathbf{r}') \approx \delta\varepsilon(\mathbf{r}) + \varepsilon_x (x' - x) + \varepsilon_y (y' - y) + \varepsilon_z (z' - z) =$$
$$= \delta\varepsilon(\mathbf{r}) + \nabla \varepsilon(\mathbf{r})(\mathbf{r}' - \mathbf{r}),$$

(symbols such as $\varepsilon_x \equiv \left.\dfrac{\partial \varepsilon(\mathbf{r}')}{\partial x'}\right|_{\mathbf{r}'=\mathbf{r}}$ are used) is presented in [6]. It has the form of

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2 \int\limits_V E_0(\mathbf{r}) \delta\varepsilon(\mathbf{r}) \frac{e^{ikR_0 \sqrt{A(\mathbf{r}, \mathbf{r}_0)}}}{4\pi R_0} d\mathbf{r}, \qquad (2)$$

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where

$$A(\mathbf{r},\mathbf{r}_0) = 1 + \delta\,\varepsilon(\mathbf{r}) + \frac{1}{2}\nabla\,\varepsilon(\mathbf{r})(\mathbf{r}_0 - \mathbf{r}). \qquad (3)$$

This article discusses ways to refine this solution.

II. The calculation of higher orders terms

Due to the complexity of the problem, we restrict ourselves to calculating the second and the third iterated kernels up to terms of the second order of expansion $\delta\varepsilon(\mathbf{r}')$ in a Taylor series

$$W_2(\mathbf{r},\mathbf{r}_0) \approx \delta\varepsilon(\mathbf{r})\int_V G(R')G(R_0')[\delta\varepsilon(\mathbf{r}) + \nabla\,\varepsilon(\mathbf{r})(\mathbf{r}' - \mathbf{r}) +$$

$$+\frac{\varepsilon_{xx}}{2}(x'-x)^2 + \frac{\varepsilon_{yy}}{2}(y'-y)^2 + \frac{\varepsilon_{zz}}{2}(z'-z)^2 +$$

$$+\varepsilon_{xy}(x'-x)(y'-y) + \varepsilon_{xz}(x'-x)(z'-z) + \varepsilon_{yz}(y'-y)(z'-z)]d\mathbf{r}',$$

where

$$\varepsilon_{xx} \equiv \left.\frac{\partial^2\,\varepsilon(\mathbf{r}')}{\partial x'^2}\right|_{\mathbf{r}'=\mathbf{r}},\ \varepsilon_{xy} \equiv \left.\frac{\partial^2\,\varepsilon(\mathbf{r}')}{\partial x'\partial y'}\right|_{\mathbf{r}'=\mathbf{r}},$$

$$R' = |\mathbf{r}' - \mathbf{r}|,\ R_0' = |\mathbf{r}' - \mathbf{r}_0|.$$

Applying the formulas for calculating the integrals [3], we get the following form of second iterated kernel

$$W_2(\mathbf{r},\mathbf{r}_0) = \frac{G(R_0)R_0\delta\,\varepsilon(\mathbf{r})}{s}\cdot$$

$$\cdot\left[\delta\varepsilon(\mathbf{r}) + \frac{\alpha_1}{2} + \frac{\alpha_2}{6} + \frac{1}{3s^2}\left(1 + \frac{sR_0}{2}\right)\beta_0\right],$$

where the designation $s = -2ik$ is used, and the functions

$$\alpha_1(\mathbf{r},\mathbf{r}_0) = \varepsilon_x(x_0 - x) + \varepsilon_y(y_0 - y) + \varepsilon_z(z_0 - z) = \nabla\,\varepsilon(\mathbf{r})(\mathbf{r}_0 - \mathbf{r}),$$

$$\alpha_2(\mathbf{r},\mathbf{r}_0) = \varepsilon_{xx}(x_0 - x)^2 + \varepsilon_{yy}(y_0 - y)^2 + \varepsilon_{zz}(z_0 - z)^2 +$$

$$+2\,\varepsilon_{xy}(x_0 - x)(y_0 - y) + 2\,\varepsilon_{xz}(x_0 - x)(z_0 - z) +$$

$$+2\,\varepsilon_{yz}(y_0 - y)(z_0 - z),$$

$$\beta_0 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \Delta\,\varepsilon(\mathbf{r})$$

describe respectively the first and second order terms of the Taylor expansion.

Similarly, the third kernel is given by

$$W_3(\mathbf{r},\mathbf{r}_0) \approx \frac{G_0(R_0)R_0\,\delta\varepsilon(\mathbf{r})}{s^2}\left\{\left(\delta\varepsilon(\mathbf{r}) + \frac{\alpha_1}{2} + \frac{\alpha_2}{6}\right)^2\left[1 + \frac{\tau}{2}\right] +\right.$$

$$\left.+\frac{2\beta_0}{s^2}\left(\delta\varepsilon(\mathbf{r}) + \alpha_1 + \frac{3\alpha_2}{10}\right)\left[1 + \frac{\tau}{2} + \frac{\tau^2}{12}\right] + \left(\frac{\nabla\,\varepsilon(\mathbf{r})}{s}\right)^2\right\}.$$

Comparing these expressions with those calculated in [6], we can argue that at least the first terms of the iterated kernels with the terms of higher orders in expansion $\delta\varepsilon(\mathbf{r}')$ taken into account, will retain their form after the function $A(\mathbf{r},\mathbf{r}_0)$ at the formula (3) is modified:

$$A(\mathbf{r},\mathbf{r}_0) = 1 + \delta\varepsilon(\mathbf{r}) + \frac{\alpha_1(\mathbf{r},\mathbf{r}_0)}{2} + \frac{\alpha_2(\mathbf{r},\mathbf{r}_0)}{6} + \frac{\alpha_3(\mathbf{r},\mathbf{r}_0)}{24} + \ldots =$$

$$= 1 + \delta\varepsilon(\mathbf{r}) + \sum_{n=1}^{\infty}\frac{\alpha_n(\mathbf{r},\mathbf{r}_0)}{(n+1)!},$$

where the functions $\alpha_n(\mathbf{r},\mathbf{r}_0)$ are the sum of all terms of the Taylor expansion of the corresponding order.

III. Further refinement of the solution

Until now, while constructing the refined solution, we have restricted ourselves to the first terms of the calculated kernels, discarding terms in the form of $(\nabla\,\varepsilon(\mathbf{r}))^2$ and higher order. Now we will consider all terms up to and including the second order. The structure of the calculated iterated kernels tells us to look for a solution in the form of

$$G(\mathbf{r},\mathbf{r}_0) = G_A(\mathbf{r},\mathbf{r}_0) +$$

$$+(\nabla A(\mathbf{r},\mathbf{r}_0))^2 G_1(\mathbf{r},\mathbf{r}_0) + \Delta A(\mathbf{r},\mathbf{r}_0)G_2(\mathbf{r},\mathbf{r}_0)\,. \qquad (4)$$

where the function $G_A(\mathbf{r},\mathbf{r}_0)$ is given by expression

$$G_A(\mathbf{r},\mathbf{r}_0) = \frac{e^{ik\sqrt{A(\mathbf{r},\mathbf{r}_0)}R_0}}{4\pi R_0},$$

and the functions $G_1(\mathbf{r},\mathbf{r}_0)$ и $G_2(\mathbf{r},\mathbf{r}_0)$ must be determined from meeting the conditions of the Helmholtz equation solution.

Substitution into the Helmholtz equation and rearrangement of the terms in the order of smallness gives

$$\Delta G + k^2\,\varepsilon(\mathbf{r})G = -G_A k^2\{A - \varepsilon + R_0\nabla A\nabla R_0\} +$$

$$+(\nabla A)^2\left\{\Delta G_1 + k^2\,\varepsilon\,G_1 - \left(\frac{ikR_0}{4A^{3/2}} + \frac{(kR_0)^2}{4A}\right)G_A\right\} +$$

$$2\nabla\left[(\nabla A)^2\right]\nabla G_1 + G_1\Delta\left[(\nabla A)^2\right] + \Delta A\left\{\Delta G_2 + k^2\,\varepsilon\,G_2 + \frac{ikR_0}{2\sqrt{A}}G_A\right\} +$$

$$+2\nabla(\Delta A)\nabla G_2 + G_2\Delta(\Delta A).$$

Therefore, to satisfy the Helmholtz equation up to second order terms, each of the curly brackets must become zero:

$$A - \varepsilon + R_0\nabla A\nabla R_0 = 0,$$

$$\Delta G_1 + k^2\,\varepsilon\,G_1 = \left(\frac{ikR_0}{4A^{3/2}} + \frac{(kR_0)^2}{4A}\right)G_A, \qquad (5)$$

$$\Delta G_2 + k^2\,\varepsilon\,G_2 = -\frac{ikR_0}{2\sqrt{A}}G_A.$$

The selection of the function $A(\mathbf{r},\mathbf{r}_0)$ ensures satisfaction of the first equation up to and including terms of at least the fourth order of smallness. The equation for the function $G_1(\mathbf{r},\mathbf{r}_0)$ is essentially the same Helmholtz equation with the known right-hand side and allows an approximate solution in the form

$$G_1(\mathbf{r},\mathbf{r}_0) = \int_V\left(\frac{ikR_0'}{4A^{3/2}(\mathbf{r}',\mathbf{r}_0)} + \frac{(kR_0')^2}{4A(\mathbf{r}',\mathbf{r}_0)}\right)\frac{e^{ikR_0'\sqrt{A(\mathbf{r}',\mathbf{r}_0)}}}{4\pi R_0'}\cdot$$

$$\frac{e^{ikR'\sqrt{A(\mathbf{r},\mathbf{r}')}}}{4\pi R'}d\mathbf{r}'$$

This solution is cumbersome and inconvenient to use, that is why we simplify it by calculating the integral approximately, assuming functions $A(\mathbf{r},\mathbf{r}')$ and $A(\mathbf{r}',\mathbf{r}_0)$ are slowly changing,

$$G_1(\mathbf{r},\mathbf{r}_0) \approx \frac{e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{32\pi ikA^{5/2}}\left[1 - ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)} - \frac{1}{3}(kR_0)^2 A(\mathbf{r},\mathbf{r}_0)\right].$$

Due to the approximation of the obtained formula for

$G_1(\mathbf{r},\mathbf{r}_0)$ there is an inaccuracy in satisfying the equation (5). Substitution the equation gives

$$\Delta G_1 + k^2\varepsilon G_1 - \left(\frac{ikR_0}{4A^{3/2}} + \frac{(kR_0)^2}{4A}\right)G_A =$$

$$-\frac{e^{ikR_0\sqrt{A}}}{32\pi ikA^{5/2}}\left\{2k^2R_0\nabla A\nabla R_0\left(1+\frac{1}{2}ikR_0\sqrt{A}\right)+\right.$$

$$+(\nabla A)^2\left[\frac{(ikR_0)^3}{3\sqrt{A}}+\frac{5(kR_0)^2}{4A}+\frac{35ikR_0}{4A^{3/2}}-\frac{35}{4A^2}-\frac{(kR_0)^4}{12}\right]+$$

$$\left.+\Delta A\left[\frac{(kR_0)^2}{6}-\frac{5ikR_0}{2\sqrt{A}}+\frac{5}{2A}\right]\right\}.$$

Similarly, solving the equation for the function $G_2(\mathbf{r},\mathbf{r}_0)$ gives

$$G_2(\mathbf{r},\mathbf{r}_0) = -\int_V \frac{ikR_0'}{2\sqrt{A(\mathbf{r}',\mathbf{r}_0)}}\frac{e^{ikR_0'\sqrt{A(\mathbf{r}',\mathbf{r}_0)}}}{4\pi R_0'}\frac{e^{ikR'\sqrt{A(\mathbf{r},\mathbf{r}')}}}{4\pi R'}d\mathbf{r}' \approx$$

$$\approx -\frac{e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{32\pi ikA^{3/2}}\left[1-ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}\right].$$

The inaccuracy in this case is

$$\Delta G_2 + k^2\varepsilon G_2 + \frac{ikR_0}{2\sqrt{A}}G_A = \frac{e^{ikR_0\sqrt{A}}}{32\pi ikA^{3/2}}\left\{k^2R_0\nabla A\nabla R_0+\right.$$

$$+(\nabla A)^2\left[\frac{(ikR_0)^3}{4\sqrt{A}}+\frac{3(kR_0)^2}{2A}+\frac{15ikR_0}{4A^{3/2}}-\frac{15}{4A^2}\right]+$$

$$\left.+\Delta A\left[\frac{3}{2A}-\frac{3ikR_0}{2\sqrt{A}}-\frac{(kR_0)^2}{2}\right]\right\}.$$

Thus, the expression for the Green's function of the Helmholtz equation, satisfying it up to and including terms of the second order of smallness, has the form of

$$G(\mathbf{r},\mathbf{r}_0)=\frac{e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{4\pi R_0}\left\{1-\frac{(\nabla A(\mathbf{r},\mathbf{r}_0))^2 ikR_0}{8k^2A^{5/2}}\right.$$

$$\left[1-ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}-\frac{1}{3}(kR_0)^2A(\mathbf{r},\mathbf{r}_0)\right]+$$

$$\left.+\frac{\Delta A(\mathbf{r},\mathbf{r}_0)ikR_0}{8k^2A^{3/2}}\left[1-ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}\right]\right\}.$$

For reference, we give the inaccuracy in satisfying the Helmholtz equation with this function

$$\Delta G + k^2\varepsilon G = \frac{-e^{ikR_0\sqrt{A}}}{32\pi ikA^{5/2}}\left[(\nabla A)^2\left\{2k^2R_0\nabla A\nabla R_0\left(1+\frac{ikR_0\sqrt{A}}{2}\right)+\right.\right.$$

$$+(\nabla A)^2\left[\frac{(ikR_0)^3}{3\sqrt{A}}+\frac{5(kR_0)^2}{4A}+\frac{35ikR_0}{4A^{3/2}}-\frac{35}{4A^2}-\frac{(kR_0)^4}{12}\right]+$$

$$\left.\Delta A\left[\frac{(kR_0)^2}{6}-\frac{5ikR_0}{2\sqrt{A}}+\frac{5}{2A}\right]\right\}-2\nabla\left[(\nabla A)^2\right]\left\{\frac{\nabla R_0}{3}k^2R_0A\right.$$

$$\left.\left(1-ikR_0\sqrt{A}\right)-\nabla A\frac{5}{2A}\left(1-ikR_0\sqrt{A}-\frac{1}{15}\left(ikR_0\sqrt{A}\right)^3\right)\right\}-$$

$$-\Delta\left[(\nabla A)^2\right]\left\{1-ikR_0\sqrt{A}+\frac{1}{3}\left(ikR_0\sqrt{A}\right)^2\right\}-\Delta A\left\{k^2R_0\nabla A\nabla R_0+\right.$$

$$+(\nabla A)^2\left[\frac{(ikR_0)^3\sqrt{A}}{4}+\frac{3(kR_0)^2}{2}+\frac{15ikR_0}{4\sqrt{A}}-\frac{15}{4A}\right]+$$

$$\left.+\Delta A\left[\frac{3}{2}-\frac{3ikR_0\sqrt{A}}{2}-\frac{(kR_0)^2A}{2}\right]\right\}+$$

$$+2\nabla(\Delta A)\left\{k^2A^2R_0\nabla R_0+\nabla A\left[\frac{(kR_0)^2A}{2}-\frac{3}{2}+\frac{3ikR_0\sqrt{A}}{2}\right]\right\}+$$

$$\left.+\Delta(\Delta A)\left\{A\left(1-ikR_0\sqrt{A}\right)\right\}\right]$$

The solution refinement process can be continued in a similar way.

The question here, in fact, is the balance between the accuracy of the formula and the complexity of the writing and the cost of obtaining it. For example, adding the terms

$$\frac{e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{4\pi R_0}\left\{(\nabla A)^3\nabla R_0\frac{R_0^3}{24A^3}\left[1+\frac{1}{2}ikR_0\sqrt{A}\right]-\right.$$

$$-\nabla\left[(\nabla A)^2\right]\nabla R_0\frac{5R_0^3}{144A^2}\left(1-\frac{2}{5}ikR_0\right)+$$

$$\left.+\Delta A\nabla A\nabla R_0\frac{R_0^3}{32A^2}-\nabla(\Delta A)\nabla R_0\frac{R_0^3}{16A}\right\}$$

to formula (4) provides accuracy up to and including terms of the third order.

IV. Conclusion

This article presents approximate solutions of various orders of accuracy to the problem of wave propagation in smooth heterogeneous medium, obtained using the iterated kernel method. The main approximation was an inaccurate consideration of the dependence of the value $\delta\varepsilon$ under the integral sign on the coordinates in the process of iterated kernels calculation. This dependence was approximately modeled by several terms of the Taylor expansion. Then, based on the summation of the Neumann series, which was carried out without additional approximations, a transition was made from the description of multiple scattering of the incident field by the in homogeneities of the medium to the description of the change in the amplitude-phase characteristics of the total field, determined by the heterogeneous medium as a whole. We note that such a transformation of the scattering process into the propagation process was carried out with mathematical rigor, in contrast to most existing asymptotic methods in which the description of wave propagation is determined by heuristic considerations based only on physical concepts.

The proposed solution has a compact form and combines the advantages of the Born scattering method and the geometric optics method, into which it passes in the extreme cases of a small and smooth change in the characteristics of a heterogeneous medium. The advantage of the proposed solution is also its applicability for any type of sounding radiation and of in homogeneities profile.

The method for increasing the accuracy of the Green's function for a heterogeneous medium by adding extra terms to its composition is shown. Their functional form is determined by the requirement of meeting the conditions of the Helmholtz equation solution and represents a compromise between the accuracy and the simplicity of the solution.

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