Riemann-Liouville fractional operators with generalized Bessel-Maitland function as its kernel

N.U.Khan* and **S.W.Khan**

Abstract- In several areas of mathematical physics and engineering sciences, integral transforms and fractional calculus operators play an important role from the application point of view. A remarkably large number of integral transforms as well as fractional integral and derivative formulas involving various special functions have been investigated by many authors. This paper is a short portrayal, concerning the utilization of Riemann-Liouville fractional operators on generalized Bessel-Maitland function. The main results demonstrate how the operators affects the parameters i.e., the Riemann-Liouville fractional integral operator and differential operator involving Bessel-Maitland function are expressed in terms of Mittag-Leffler functions. The main results can be applied to obtain certain special cases by specialising the parameters.

Keywords- Generalized Bessel-Maitland function, generalized (Wright) hypergeometric functions $_p\Psi_q$, Riemann-Liouville fractional operators.

1 Introduction

1.1 Bessel-Maitland function

In Applied sciences, several vital functions are outlined via improper integrals or series (or finite products). The general name of these vital functions is aware of as special functions. In special functions, one amongst the foremost vital function (Bessel function) is widely utilized in physics and engineering, they are of interest to physicists and engineers in addition as man of science. In neoteric years a remarkably sizable amount of integral formulas involving a range of special functions are developed by several authors (see [1],[5]-[8],[14])

For our motive, we start by retracing certain famous functions and supra works. The Bessel-Maitland function $J_\nu^\mu(z)$ outlined by the following series illustration Merichev [9] as follows:

$$J_\nu^\mu(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\nu+\mu n+1)n!},\qquad \mu>0; z\in\mathbb{C} \tag{1.1}$$

The generalized Bessel function of the form $J_{\nu,\sigma}^\mu(z)$ is defined by Jain and Agarwal [11] as follows:

$$J_{\nu,\sigma}^\mu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{\nu+2\sigma+2n}}{\Gamma(\nu+\sigma+\mu n+1)\Gamma(\sigma+n+1)} \tag{1.2}$$

where $z\in\mathbb{C}\setminus(-\infty,0], \mu>0, \nu,\sigma\in\mathbb{C}$.

Further, generalization of the generalized Bessel-Maitland function, $J_{\nu,q}^{\mu,\gamma}(z)$ defined by Pathak [10] as follows:

$$J_{\nu,q}^{\mu,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma(\nu+\mu n+1)n!} \tag{1.3}$$

where $\mu,\nu,\gamma\in\mathbb{C}, \Re(\mu)\geq 0, \Re(\nu)\geq -1, \Re(\gamma)\geq 0$ *and* $q\in(0,1)\cup\mathbb{N}$. From the generalization of the Bessel-Maitland function (1.3), it is possible to find find relations between Bessel-Maitland function and Mittag Leffler function.

If ν is replaced by $\nu-1$ and z by $-z$, (1.3) reduces to

$$J_{\nu-1,q}^{\mu,\gamma}(-z)=E_{\mu,\nu}^{\gamma;q}(z) \tag{1.4}$$

where $\mu,\nu,\gamma\in\mathbb{C};\Re(\mu)>0,\Re(\nu)>0,\Re(\gamma)>0; q\in(0,1)\cup\mathbb{N}$ and $E_{\mu,\nu}^{\gamma;q}(z)$ denotes generalized Mittag-Leffler function, was given by Shukla and Prajapati [3]

If $q=1$,ν is replaced by $\nu-1$ and z by $-z$, (1.3) reduces to

$$J_{\nu-1,1}^{\mu,\gamma}(-z)=E_{\mu,\nu}^{\gamma}(z) \tag{1.5}$$

where $\mu, \nu, \gamma \in \mathbb{C}; \Re(\mu) > 0, \Re(\nu) > 0, \Re(\gamma) > 0$ and $E^{\gamma}_{\mu,\nu}(z)$ was introduced by Prabhakar [13]
If $q = 1$, $\gamma = 1$, ν is replaced by $\nu - 1$ and z by $-z$, (1.3) reduces to

$$J^{\mu,1}_{\nu-1,1}(-z) = E_{\mu,\nu}(z) \tag{1.6}$$

where $\mu \in \mathbb{C}$, $\Re(\mu) > 0, \Re(\nu) > 0$, was studied by Wiman [4]

1.2 Fractional derivative and integral operators

Differentiation and integration of fractional order are traditionally defined by the right sided Riemann-Liouville fractional integral operator I^{μ}_{a+} and left sided Riemann-Liouville fractional integral operator I^{μ}_{a-} and the corresponding Riemann-Liouville fractional derivative operator D^{μ}_{a+} and D^{μ}_{a-} as follows [12, p-33(2.17,2.18),p-37(2.32,2.33)]

$$\left(I^{\mu}_{a+}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_a^x \frac{f(t)}{(x-t)^{1-\mu}}dt, \tag{1.7}$$

where $(x > a; \Re(\mu) > 0)$.

$$\left(I^{\mu}_{a-}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_x^a \frac{f(t)}{(t-x)^{1-\mu}}dt, \tag{1.8}$$

where $(x < a; \Re(\mu) > 0)$.

$$\left(D^{\mu}_{a+}f\right)(x) = \frac{d^n}{dx^n}\left(I^{n-\mu}_{a+}f\right)(x), \tag{1.9}$$

where $(\Re(\mu) \geq 0, n = 1 + [\Re(\mu)])$.

$$\left(D^{\mu}_{a-}f\right)(x) = (-1)^n\frac{d^n}{dx^n}\left(I^{n-\mu}_{a+}f\right)(x), \tag{1.10}$$

where $(\Re(\mu) \geq 0, n = 1 + [\Re(\mu)])$.
In above equations the function f is locally integrable, $\Re(\mu)$ denotes real part of the complex number and$[\Re(\mu)]$ means greatest integer in $\Re(\mu)$.
We will need the following result [2, p-10(13)]

$$\int_b^a (a-t)^{\beta-1}(t-b)^{\alpha-1}dt = (a-b)^{\alpha+\beta-1}B(\alpha,\beta), \tag{1.11}$$

where $(\Re(\alpha) > 0, \Re(\beta) > 0, b < a)$

2 Fractional Integral of generalized Bessel-Maitland function

In this section we are going to discuss the result concerning the Bessel-Maitland function under the Riemann-Liouville fractional integral operator.

Theorem 2.1. *The following integral formula holds :*

$$\begin{aligned}&\left(I^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x)\\ &= (x-a)^{\nu+\lambda}J^{\mu,\gamma}_{\nu+\lambda,q}(\omega(x-a)^{\mu})\end{aligned} \tag{2.1}$$

where $x > a; \omega, \lambda, \mu, \nu, \gamma \in \mathbb{C}; q \in (0,1) \cup \mathbb{N}; \Re(\mu) \geq 0; \Re(\gamma) \geq 0; \Re(\nu) \geq -1; \Re(\lambda) > 0$

Proof. By using (1.3) and the definition of integral operator in (1.7) and interchanging integral and summation, which is verified under uniform convergence of series, we get

$$\left(I^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x) = \frac{1}{\Gamma(\lambda)}$$

$$\times\sum_{m=0}^{\infty}\frac{(\gamma)_{qn}(-\omega)^m}{m!\Gamma(\nu+\mu m+1)}\int_a^x (x-t)^{\lambda-1}(t-a)^{\nu+\mu m}dt$$

Now using (1.11), we get

$$\left(I^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x) = (x-a)^{\nu+\lambda}$$

$$\times\sum_{m=0}^{\infty}\frac{(\gamma)_{qn}(-\omega)^m(x-a)^{\mu m}}{\Gamma(\nu+\lambda+\mu m+1)}$$

The required result is obtained by using (1.3). □

Corollary 2.2. *If ν is replaced by $\nu - 1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (2.1), then using (1.4), we get :*

$$\left(I^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,\gamma}_{\nu-1,q}(-\omega(t-a)^{\mu})\right)(x) = (x-a)^{\nu+\lambda-1}$$

$$\times E^{\gamma,q}_{\mu,\nu+\lambda}(\omega(x-a)^{\mu})$$

Corollary 2.3. *If q=1, ν is replaced by $\nu - 1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (2.1), then using (1.5), we get :*

$$\left(I^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,\gamma}_{\nu-1,1}(-\omega(t-a)^{\mu})\right)(x) = (x-a)^{\nu+\lambda-1}$$

$$\times E^{\gamma}_{\mu,\nu+\lambda}(\omega(x-a)^{\mu})$$

Corollary 2.4. *If q=1, $\gamma = 1$,ν is replaced by $\nu - 1$ and $\omega(t-a)^\mu$ by $-\omega(t-a)^\mu$ in L.H.S of (2.1), then using (1.6), we get :*

$$\left(I^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,1}_{\nu-1,1}(-\omega(t-a)^{\mu})\right)(x) = (x-a)^{\nu+\lambda-1} \times E_{\mu,\nu+\lambda}\left(\omega(x-a)^{\mu}\right)$$

In the above corollaries we must have

$$x > a; \omega, \lambda, \mu, \nu, \gamma \in \mathbb{C}; q \in (0,1) \cup \mathbb{N};$$

$$\Re(\mu) > 0; \Re(\gamma) > 0; \Re(\nu) > 0; \Re(\lambda) > 0$$

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Theorem 2.5. *The following integral formula holds :*

$$\left(I^{\lambda}_{a-}(a-t)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(a-t)^{\mu})\right)(x) = (a-x)^{\nu+\lambda} \times J^{\mu,\gamma}_{\nu+\lambda,q}(\omega(a-x)^{\mu}) \quad (2.2)$$

where $x < a; \omega, \lambda, \mu, \nu, \gamma \in \mathbb{C}; q \in (0,1) \cup \mathbb{N}; \Re(\mu) \geq 0; \Re(\gamma) \geq 0; \Re(\nu) \geq -1; \Re(\lambda) > 0$

Proof. The above result can be obtained on similar steps as in Theorem 2.1 . □

Corollary 2.6. *If ν is replaced by $\nu - 1$ and $\omega(t-a)^\mu$ by $-\omega(t-a)^\mu$ in L.H.S of (2.2), then using (1.4), we get :*

$$\left(I^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,\gamma}_{\nu-1,q}(-\omega(a-t)^{\mu})\right)(x) = (a-x)^{\nu+\lambda-1}E^{\gamma,q}_{\mu,\nu+\lambda}\left(\omega(a-x)^{\mu}\right)$$

Corollary 2.7. *If q=1, ν is replaced by $\nu - 1$ and $\omega(t-a)^\mu$ by $-\omega(t-a)^\mu$ in L.H.S of (2.2), then using (1.5), we get :*

$$\left(I^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,\gamma}_{\nu-1,1}(-\omega(a-t)^{\mu})\right)(x) = (a-x)^{\nu+\lambda-1}E^{\gamma}_{\mu,\nu+\lambda}\left(\omega(a-x)^{\mu}\right)$$

Corollary 2.8. *If q=1, $\gamma = 1$,ν is replaced by $\nu - 1$ and $\omega(t-a)^\mu$ by $-\omega(t-a)^\mu$ in L.H.S of (2.2), then using (1.6), we get :*

$$\left(I^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,1}_{\nu-1,1}(-\omega(a-t)^{\mu})\right)(x) = (a-x)^{\nu+\lambda-1}E_{\mu,\nu+\lambda}\left(\omega(a-x)^{\mu}\right)$$

In the above corollaries we must have

$$x < a; \omega, \lambda, \mu, \nu, \gamma \in \mathbb{C}; q \in (0,1) \cup \mathbb{N};$$

$$\Re(\mu) > 0; \Re(\gamma) > 0; \Re(\nu) > 0; \Re(\lambda) > 0$$

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3 Fractional Derivative of generalized Bessel-Maitland function

In this section we are going to discuss the results concerning the Bessel-Maitland function under the Riemann-Liouville fractional differential operator.

Theorem 3.1. *The following integral formula holds*

$$\left(D^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x) = (x-a)^{\nu-\lambda} \times J^{\mu,\gamma}_{\nu-\lambda,q}(\omega(x-a)^{\mu}) \quad (3.1)$$

where $x > a; \omega, \lambda, \mu, \nu, \gamma \in \mathbb{C}; q \in (0,1) \cup \mathbb{N}; \Re(\mu) \geq 0; \Re(\gamma) \geq 0; \Re(\nu) \geq -1; \Re(\lambda) \geq 0$

Proof. By using (1.3) and the definition of integral operator in (1.9) and interchanging integral and summation, which is verified under uniform convergence of series, we get

$$\left(D^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x) = \frac{1}{\Gamma(n-\lambda)}$$

$$\times \sum_{m=0}^{\infty} \frac{(\gamma)_{qn}(-\omega)^m}{m!\Gamma(\nu+\mu m+1)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\lambda-1}(t-a)^{\nu+\mu m}dt$$

Now using (1.11), we get

$$\left(D^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x)$$

$$= \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(-\omega)^m}{m!\Gamma(\nu-\lambda+\mu m+n+1)} \frac{d^n}{dx^n}(x-a)^{\nu-\lambda+n+\mu m}$$

Now using

$$\frac{d^n}{dx^n}(x^a) = \frac{\Gamma(a+1)}{\Gamma(a+1-m)}x^{a-m}, \quad \textit{where } \Re(a) > 0 \textit{ we get}$$

$$\left(D^{\lambda}_{a+}(t-a)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(t-a)^{\mu})\right)(x)$$

$$= (x-a)^{\nu-\lambda} \sum_{m=0}^{\infty} \frac{(\gamma)_{qn}(-\omega)^m(x-a)^{\mu m}}{\Gamma(\nu-\lambda+\mu m+1)}$$

which upon using (1.3) gives the required result . □

Corollary 3.2. *If ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.1), then using (1.4), we get :*

$$\left(D^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,\gamma}_{\nu-1,q}(-\omega(t-a)^{\mu})\right)(x)$$
$$=(x-a)^{\nu-\lambda-1}E^{\gamma,q}_{\mu,\nu-\lambda}\left(\omega(x-a)^{\mu}\right)$$

Corollary 3.3. *If q=1, ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.1), then using (1.5), we get :*

$$\left(D^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,\gamma}_{\nu-1,1}(-\omega(t-a)^{\mu})\right)(x)$$
$$=(x-a)^{\nu-\lambda-1}E^{\gamma}_{\mu,\nu-\lambda}\left(\omega(x-a)^{\mu}\right)$$

Corollary 3.4. *If q=1, $\gamma=1$,ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.1), then using (1.6), we get :*

$$\left(D^{\lambda}_{a+}(t-a)^{\nu-1}J^{\mu,1}_{\nu-1,1}(-\omega(t-a)^{\mu})\right)(x)$$
$$=(x-a)^{\nu-\lambda-1}E_{\mu,\nu-\lambda}\left(\omega(x-a)^{\mu}\right)$$

In the above corollaries we must have

$$x>a;\omega,\lambda,\mu,\nu,\gamma\in\mathbb{C};q\in(0,1)\cup\mathbb{N};$$

$$\Re(\mu)>0;\Re(\gamma)>0;\Re(\nu)>0;\Re(\lambda)>0$$

.

Theorem 3.5. *The following integral formula holds*

$$\left(D^{\lambda}_{a-}(a-t)^{\nu}J^{\mu,\gamma}_{\nu,q}(\omega(a-t)^{\mu})\right)(x)=(a-x)^{\nu-\lambda}$$
$$\times J^{\mu,\gamma}_{\nu-\lambda,q}(\omega(a-x)^{\mu}) \qquad (3.2)$$

where $x<a;\omega,\lambda,\mu,\nu,\gamma\in\mathbb{C};q\in(0,1)\cup\mathbb{N};\Re(\mu)\geq 0;\Re(\gamma)\geq 0;\Re(\nu)\geq -1;\Re(\lambda)\geq 0$

Proof. The above result can be proved on similar steps as in Theorem 3.1 . □

Corollary 3.6. *If ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.2), then using (1.4), we get :*

$$\left(D^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,\gamma}_{\nu-1,q}(-\omega(a-t)^{\mu})\right)(x)$$
$$=(a-x)^{\nu-\lambda-1}E^{\gamma,q}_{\mu,\nu-\lambda}\left(\omega(a-x)^{\mu}\right)$$

Corollary 3.7. *If q=1, ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.2), then using (1.5), we get :*

$$\left(D^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,\gamma}_{\nu-1,1}(-\omega(a-t)^{\mu})\right)(x)$$
$$=(a-x)^{\nu-\lambda-1}E^{\gamma}_{\mu,\nu-\lambda}\left(\omega(a-x)^{\mu}\right)$$

Corollary 3.8. *If q=1, $\gamma=1$,ν is replaced by $\nu-1$ and $\omega(t-a)^{\mu}$ by $-\omega(t-a)^{\mu}$ in L.H.S of (3.2), then using (1.6), we get :*

$$\left(D^{\lambda}_{a-}(a-t)^{\nu-1}J^{\mu,1}_{\nu-1,1}(-\omega(a-t)^{\mu})\right)(x)$$
$$=(a-x)^{\nu-\lambda-1}E_{\mu,\nu-\lambda}\left(\omega(a-x)^{\mu}\right)$$

In the above corollaries we must have

$$x>a;\omega,\lambda,\mu,\nu,\gamma\in\mathbb{C};q\in(0,1)\cup\mathbb{N};$$

$$\Re(\mu)>0;\Re(\gamma)>0;\Re(\nu)>0;\Re(\lambda)>0$$

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N. U. Khan*
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh, 202002, India.
e-mail: nukhanmath@gmail.com

S. W. Khan
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh, 202002, India.
e-mail: swalikhan6@gmail.com