Method of iterated kernels in problems of wave propagation in heterogeneous media

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***Abstract*—The approximated solution of the wave propagation problem in smoothly heterogeneous medium on the basis of the method of iterated kernels is proposed in this article. The solution is obtained by the method of successive approximations to an integral equation, which is equivalent to the Helmholtz scalar equation. Since the exact calculation of iterated kernels is impossible for arbitrary spatial dependence of medium dielectric permittivity, approximate estimation is used applying several first Taylor expansion terms. In purpose of exact calculating of the double series for resolvent a method, based on identifying of coefficients of a power series with orthogonal polynomial, which is calculated by Rodrig's generalized formula, will be applied. The final solution has a compact form and unites the advantages of Born scattering and short-wave asymptotic methods. The proposed solution requires smoothness of medium heterogeneities changes, scilicet the smallness of first and second derivatives of the dielectric permittivity, but not of the dielectric permittivity itself.**

***Keywords*—wave propagation, method of iterated kernels, Born approximation, the smooth perturbations method.**

I. Introduction

The problem of wave propagation in various media, as well as the problems of their generation and receiving, is foundational for acoustic, radiophysics and optics. So far a lot of approximated methods in theory of wave propagation have been developed. They can be divided into two big classes: methods describing wave scattering and methods considering wave propagation.

The first class methods include the theory of single scattering, the Twersky theory, the theory of multiple propagations, the Dyson equation, the radiative transfer equation, etc. [1, 2]. These methods describe the process of wave propagation in medium with small-scale heterogeneities in comparison with the wavelength, which is followed by formation of secondary radiation as the result of scattering. Whereby the characteristic changes of the incident wave are not taken into account.

The second class of methods is formed by asymptotic methods, which describe mostly the phase change of the incident wave due to passing through the medium with large-scale heterogeneities. Whereby it is considered that scattering occurs basically in the direction of propagation of original wave and it can be neglected. The geometric optics method, the smooth perturbations method (the Rytov approximation) and the method of parabolic equation [1, 2] are the most important methods of this approach.

The problem of creation of effective method for medium with different scales of heterogeneities, which would describe both a scattered field and the incident wave distortion has not been solved yet. For this reason, describing interaction between a wave and medium heterogeneities only one of the dominant effects is used – scattering, absorption, refraction, diffraction etc. – the other aren’t taken into account.

In the article the approximate solution of the Helmholtz equation, based on the applying of iterated kernels method to equivalent integral equation, is proposed. Since the exact calculation of iterated kernels is impossible for arbitrary spatial dependence of medium dielectric permittivity, approximate estimation is used applying several first Taylor expansion terms. The obtained series of iterated kernels can be summarized precisely, that leads to rather adequate for analysis result.

II. Method of iterated kernels

We will consider the problem of radiowave propagation in heterogeneous borderless medium in scalar case. As known, the problem of total electric field finding consist in solving of inhomogeneous integral Fredholm equation of second kind.

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2 \int_V G_0(R_0) E(\mathbf{r}) \delta\varepsilon(\mathbf{r}) d\mathbf{r}. \qquad (1)$$

Here the $E_0(\mathbf{r}_0)$ is primary wave electric field strength, $\delta\varepsilon(\vec{r}) = (\varepsilon(\vec{r}) - \varepsilon_0)/\varepsilon_0$ is the disturbed value of medium dielectric permittivity towards the background value ε_0, $G_0(R_0) = \dfrac{e^{ikR_0}}{4\pi R_0}$ is Green’s function of infinite homogeneous medium, $R_0 = |\mathbf{r}_0 - \mathbf{r}|$. The integral along unbounded space volume V in the right part represents the field scattered by heterogeneities. Functions $E(\mathbf{r}_0), E_0(\mathbf{r}_0), \varepsilon(\mathbf{r})$ relate to the space of square integrable functions.

It can be shown, that by the notion of resolvent the following solution for the integral equation can be proposed

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2 \int_V E_0(\mathbf{r}) \Gamma(\mathbf{r}, \mathbf{r}_0) d\mathbf{r}, \qquad (2)$$

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where $\Gamma(\mathbf{r},\mathbf{r}_0)$ is an integral equation (1) resolvent.

Under arbitrary spatial dependence of dielectric permittivity contrast $\delta\varepsilon(\mathbf{r})$ a basic method of solution for the integral equation (1) is the method of iterated kernels. In case of linear medium the resolvent can be presented by a Neumann series

$$\Gamma(\mathbf{r},\mathbf{r}_0) = \sum_{n=0}^{\infty} k^{2n} W_{n+1}(\mathbf{r},\mathbf{r}_0), \qquad (3)$$

which converges in case of sufficiently small values of the wave number k. *n+1*-st iterated kernel $W_{n+1}(\mathbf{r},\mathbf{r}_0)$ can be found by the following recurrent relation [3]

$$W_{n+1}(\mathbf{r},\mathbf{r}_0) = \int W_n(\mathbf{r},\mathbf{r}')W(\mathbf{r}',\mathbf{r}_0)d\mathbf{r}', \qquad (4)$$

where $W_1(\mathbf{r},\mathbf{r_0}) = W(\mathbf{r},\mathbf{r_0})$ is the kernel of integral equation (1).

The main difficulty of such approach lies in the cumbersomeness of iterated kernels writing as they represent multidimensional integrals of rather complicated form that can’t be summarized. Thus it is necessary to either be limited to a small quantity of considered kernels (Born approximation, double scattering theory [2], etc.) or to use simplifying approximations.

We are coming now to the calculation of iterated kernels. In our case the first kernel equals to

$$W_1(\mathbf{r},\mathbf{r}_0) = G_0(R_0)\,\delta\varepsilon(\mathbf{r}).$$

The second iterated kernel can be found by the formula (4)

$$W_2(\mathbf{r},\mathbf{r}_0) = \delta\varepsilon(\mathbf{r})\int_V \delta\varepsilon(\mathbf{r}')G_0(R_0')G_0(R')d\mathbf{r}',$$

where $R_0' = |\mathbf{r}_0 - \mathbf{r}'|$, $R' = |\mathbf{r} - \mathbf{r}'|$. On the assumption of sufficiently smooth change of $\delta\varepsilon(\mathbf{r}')$ the first multiplier in integrand can be decomposed into a Taylor series about the point **r**, restricting to its linear terms:

$$\delta\varepsilon(\mathbf{r}') \approx \delta\varepsilon(\mathbf{r}) + \varepsilon_x(x'-x) + \varepsilon_y(y'-y) + \varepsilon_z(z'-z) + \ldots = \\ = \delta\varepsilon(\mathbf{r}) + \nabla\varepsilon(\mathbf{r})(\mathbf{r}'-\mathbf{r}) + \ldots$$

where designations $\varepsilon_x \equiv \left.\frac{\partial\varepsilon(\mathbf{r}')}{\partial x'}\right|_{\mathbf{r}'=\mathbf{r}}$ and etc. were used.

The absence of the arbitrary function under the integral allows to calculate the value of the integral precisely [4]:

$$W_2(\mathbf{r},\mathbf{r}_0) \approx \delta\varepsilon(\mathbf{r})\int_V G_0(R')G_0(R_0')[\delta\varepsilon(\mathbf{r}) + \nabla\varepsilon(\mathbf{r})(\mathbf{r}'-\mathbf{r})]d\mathbf{r}' = \\ = \frac{G_0(R_0)R_0\Delta\varepsilon(\mathbf{r})}{s}\left[\delta\varepsilon(\mathbf{r}) - \frac{1}{2}\nabla\varepsilon(\mathbf{r})(\mathbf{r}-\mathbf{r}_0)\right].$$

With usage of the designations

$$\alpha(\mathbf{r},\mathbf{r}_0) = \nabla\varepsilon(\mathbf{r})(\mathbf{r}_0 - \mathbf{r}),$$

$$\delta A(\mathbf{r},\mathbf{r}_0) = \delta\varepsilon(\mathbf{r}) + \frac{1}{2}\alpha(\mathbf{r},\mathbf{r}_0), \quad s = -2ik$$

this solution can be set out in a compact form

$$W_2(\mathbf{r},\mathbf{r}_0) \approx \frac{G_0(R_0)R_0}{s}\delta\varepsilon(\mathbf{r})\delta A(\mathbf{r},\mathbf{r}_0).$$

The other iterated kernels are calculated similarly. As an example several first iterated kernels are listed

$$W_3(\mathbf{r},\mathbf{r}_0) \approx \frac{G_0(R_0)R_0\,\delta\varepsilon(\mathbf{r})}{s^3}\left\{(\delta\mathrm{A})^2\left[1+\frac{\tau}{2}\right] + \frac{\beta}{s^2}\left[1+\frac{\tau}{2}+\frac{\tau^2}{12}\right]\right\},$$

$$W_4 \approx \frac{G_0(R_0)R_0\delta\varepsilon(\mathbf{r})}{s^5}\cdot \\ \cdot\left\{2(\delta A)^3\left[1+\frac{\tau}{2}+\frac{\tau^2}{12}\right] + \frac{\beta}{s^2}\left[1+\frac{\tau}{2}+\frac{\tau^2}{10}+\frac{\tau^3}{120}\right](10\delta A + 3(\gamma-\alpha))\right\},$$

$$W_5(\mathbf{r},\mathbf{r}_0) \approx \frac{G_0(R_0)R_0\delta\varepsilon(\mathbf{r})}{s^7}\left\{5(\delta A)^4\left[1+\frac{\tau}{2}+\frac{\tau^2}{10}+\frac{\tau^3}{120}\right] + \right. \\ + \frac{\beta}{s^2}\left[1+\frac{\tau}{2}+\frac{3\tau^2}{28}+\frac{\tau^3}{84}+\frac{\tau^4}{1680}\right]\cdot \\ \cdot\left(70(\delta A)^2 + 7(\gamma-\alpha)\delta A + 16.5\alpha^2 + 2.5\alpha\gamma\right) + \\ \left. + 204\left(\frac{\beta}{s^2}\right)^2\left[1+\frac{\tau}{2}+\frac{\tau^2}{9}+\frac{\tau^3}{72}+\frac{\tau^4}{1008}+\frac{\tau^5}{30240}\right]\left(1+\frac{3\delta}{17}\right)\right\},$$

where designations are used

$$\tau = -2ikR_0, \ \beta(\mathbf{r},\mathbf{r}_0) = \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 = (\nabla\varepsilon)^2,$$

$$\gamma(\mathbf{r},\mathbf{r}_0) = \frac{\varepsilon_x^3(x_0-x) + \varepsilon_y^3(y_0-y) + \varepsilon_z^3(z_0-z)}{\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2},$$

$$\delta(\mathbf{r},\mathbf{r}_0) = \frac{\varepsilon_x^4+\varepsilon_y^4+\varepsilon_z^4}{\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2\right)^2}.$$

The first calculated iterated kernels already demonstrate very complex structure of the problem’s exact solution: all expansion terms in powers of $\nabla\varepsilon$ are present in it. Consideration of the further expansion terms of Taylor series will add components proportional to derivatives of $\delta\varepsilon$ of second, third and all the following degrees in various combinations into the solution. At the same time first summands of every calculated kernel comply with the common pattern.

$$W_{n+1}(\mathbf{r},\mathbf{r}_0) = G_0(R_0)\delta\varepsilon(\mathbf{r})[\delta A(\mathbf{r})]^n \frac{ikR_0}{s^{2n}}\sum_{m=0}^{n-1}\frac{a_{m,n}\tau^m}{(m+1)!}, \qquad (5)$$

where coefficients $a_{m,n}$ satisfy the recurrence relations

$$a_{0,n} = \sum_{i=0}^{n-2} a_{i,n-1}, \ a_{n-1,n} = 2,$$

$$a_{m,n} = \sum_{i=m-1}^{n-2} a_{i,n-1}, \ m = 1,2,\ldots n-2$$

After insertion (5) into (3) the resolvent will be written as

$$\Gamma(\mathbf{r},\mathbf{r}_0) = \delta\varepsilon(\mathbf{r})G_0(R_0)\left(1 - ikR_0\sum_{n=1}^{\infty}\left[-\frac{\delta A(\mathbf{r})}{4}\right]^n\sum_{m=0}^{n-1}\frac{a_{m,n}\tau^m}{(m+1)!}\right)$$

In purpose of calculating this double series a method, proposed in [4], will be applied. This method is based on identifying of coefficients of a power series with orthogonal polynomial, which is calculated by Rodrig’s generalized formula [5]. As a result it comes to the following compact resolvent representation

$$\Gamma(\mathbf{r},\mathbf{r}_0) = \frac{\delta\varepsilon(\mathbf{r})e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{4\pi R_0},$$

where $A(\mathbf{r},\mathbf{r}_0) = 1 + \delta A(\mathbf{r},\mathbf{r}_0)$.

Thus solution of the equation (1) will be written as

$$E(\mathbf{r}_0) = E_0(\mathbf{r}_0) + k^2\int_V E_0(\mathbf{r})\delta\varepsilon(\mathbf{r})\frac{e^{ikR_0\sqrt{A(\mathbf{r},\mathbf{r}_0)}}}{4\pi R_0}d\mathbf{r}. \qquad (6)$$

Let us choose as incident field the point source field $E_0(\mathbf{r},\mathbf{r}_1)=G_0(R_1)$. After that evaluation of the integral will give Green's function of Helmholtz equation for heterogeneous medium

$$G_A(\mathbf{r_0},\mathbf{r_1})=\frac{e^{ik\sqrt{A(\mathbf{r}_0,\mathbf{r}_1)}R_{01}}}{4\pi R_{01}},\ R_{01}=|\mathbf{r_0}-\mathbf{r_1}|. \quad (7)$$

Let us emphasize that the only approximation made during the conversion process consist in inaccurate consideration of coordinate dependence $\delta\varepsilon$. The proposed solution combines the benefits of Born scattering and geometrical optics methods. In a point of fact, in case of small fluctuations of dielectric permittivity, when influence of $\delta\varepsilon$ in the exponential function can be neglected, our solution coincides with Born's formula. On the other hand, exponential factor, which describes the distortion of the wave that passed through heterogeneous medium, can be equated with optical length which forms the basis of geometrical optics method. Dignity of the proposed method lies in its satisfiability for any kind of incident wave and heterogeneity profile.

III. Accuracy estimation and comparison between other methods

To determine the suggested approach place among other approximate methods of wave propagation in heterogeneous medium let's estimate accuracy of the solution (6) and the most well-known methods - Born approximation and method of smooth perturbations.

Accuracy valuation problem and boundaries establishment for satisfiability of various approximate methods very likely remains the most obscure and disputable topic of wave propagation theory [1, 2]. Taking that into account, the disparity of approximate solution, substituted to Helmholtz equation, will be considered as a criterion of accuracy. To keep uniformity in every case the point-source field (Green's function) will be reviewed.

Let's begin with Born approximation

$$G_B(\mathbf{r},\mathbf{r}_0)=G_0(R_0)+k^2\int_V \delta\varepsilon(\mathbf{r}')G_0(R')G_0(R_0')d\mathbf{r}'. \quad (8)$$

Insertion in Helmholtz equation leads to expression for disparity (when $\mathbf{r}\neq\mathbf{r}_0$)

$$N[G_B(\mathbf{r},\mathbf{r}_0)]\equiv\Delta G_B(\mathbf{r},\mathbf{r}_0)+k^2\varepsilon(\mathbf{r})G_B(\mathbf{r},\mathbf{r}_0)= \\ =k^2\delta\varepsilon(\mathbf{r})G_B(\mathbf{r},\mathbf{r}_0).$$

Using the approach already used in calculating of iterative kernels [4], evaluation of the integral in (8) gives

$$G_B(\mathbf{r},\mathbf{r}_0)=G_0(R_0)\cdot \\ \cdot\left[1+\frac{ikR_0\delta A(\mathbf{r},\mathbf{r}_0)}{2}+\frac{R_0\Delta\varepsilon(\mathbf{r})}{24ik}(1-ikR_0)+...\right]. \quad (9)$$

Consequently, Born approximation usability condition is first of all defined by requirement of smallness of heterogeneities of permeability contrast $\delta\varepsilon(\mathbf{r})$ in relation to background environment.

Green's function in method of smooth perturbations approximation has form of [2]

$$G_R(\mathbf{r},\mathbf{r}_0)=G_0(R_0)e^{k^2S(\mathbf{r},\mathbf{r}_0)}, \quad (10)$$

where

$$S(\mathbf{r},\mathbf{r}_0)=\int_V \delta\varepsilon(\mathbf{r}')\frac{G_0(R')G_0(R_0')}{G_0(R_0)}d\mathbf{r}'.$$

Insertion in Helmholtz equation leads to expression for disparity

$$N[G_R(\mathbf{r},\mathbf{r}_0)]=k^2G_R(\mathbf{r},\mathbf{r}_0)\Big[\delta\varepsilon(\mathbf{r})+k^2(\nabla S(\mathbf{r},\mathbf{r}_0))^2+ \\ +2\left(ik-\frac{1}{R_0}\right)\nabla R_0\nabla S(\mathbf{r},\mathbf{r}_0)+\Delta S(\mathbf{r},\mathbf{r}_0)\Big].$$

Let's calculate derivatives of first and second order from $S(\mathbf{r},\mathbf{r}_0)G_0(R_0)$:

$$\nabla[S(\mathbf{r},\mathbf{r}_0)G_0(R_0)]=G_0(R_0)\nabla S(\mathbf{r},\mathbf{r}_0)+S(\mathbf{r},\mathbf{r}_0)\nabla G_0(R_0)= \\ =\int_V \delta\varepsilon(\mathbf{r}')G_0(R')G_0(R_0')\left(ik-\frac{1}{R'}\right)\nabla R'd\mathbf{r}',$$

$$\Delta[S(\mathbf{r},\mathbf{r}_0)G_0(R_0)]==2\nabla G_0(R_0)\nabla S(\mathbf{r},\mathbf{r}_0)+G_0(R_0)\Delta S(\mathbf{r},\mathbf{r}_0)+ \\ +S(\mathbf{r},\mathbf{r}_0)\Delta G_0(R_0)= \\ =\int_V \delta\varepsilon(\mathbf{r}')G_0(R')G_0(R_0')\left[\left(ik-\frac{1}{R'}\right)^2+2\left(ik-\frac{1}{R'}\right)\frac{1}{R'}+\frac{1}{R'^2}\right]d\mathbf{r}'= \\ =-k^2S(\mathbf{r},\mathbf{r}_0)G_0(R_0).$$

Therefore

$$2\nabla G_0(R_0)\nabla S(\mathbf{r},\mathbf{r}_0)+G_0(R_0)\Delta S(\mathbf{r},\mathbf{r}_0)=0,$$

and disparity of smooth perturbations method describes with relator

$$N[G_R(\mathbf{r},\mathbf{r}_0)]=k^2G_R(\mathbf{r},\mathbf{r}_0)\Big[\delta\varepsilon(\mathbf{r})+k^2(\nabla S(\mathbf{r},\mathbf{r}_0))^2\Big].$$

After approximate evaluation of the integral for $S(\mathbf{r},\mathbf{r}_0)$

$$S(\mathbf{r},\mathbf{r}_0)=-\frac{R_0}{2ik}\left[\delta A(\mathbf{r},\mathbf{r}_0)-\frac{1}{12k^2}(1-ikR_0)\Delta\varepsilon(\mathbf{r})+...\right] \quad (11)$$

the disparity dominant term describes with expression

$$N[G_R(\mathbf{r},\mathbf{r}_0)]\approx k^2G_R(\mathbf{r},\mathbf{r}_0)\left[\delta\varepsilon(\mathbf{r})-\frac{1}{4}(\delta\varepsilon(\mathbf{r}))^2+...\right].$$

Hence, the requirement of smallness of medium heterogeneities contrast $\delta\varepsilon(\mathbf{r})$ is also a usability condition for the method of smooth perturbations. This result turns up sudden whereas it is usually believed that the method is destined for characterization of smooth medium for which the permittivity gradient is small but not the permittivity itself. Let's stress that in contrast with Born approximation in the Rytov's method the dominant term of inaccuracy does not depend on value kR_0, which allows to expand the calculation results on the domain of large distances.

Now let's research the disparity of Helmholtz equation for solution (7), proposed in this article.

$$N[G_A(\mathbf{r},\mathbf{r}_0)]=-G_A\Big\{k^2(A-\varepsilon+R_0\nabla A\nabla R_0)+ \\ +(\nabla A)^2\left(\frac{ikR_0}{4A^{3/2}}+\frac{(kR_0)^2}{4A}\right)-\frac{ikR_0}{2\sqrt{A}}\Delta A\Big\}.$$

Expression of disparity by means of $\varepsilon(\mathbf{r})$ and its derivatives selects the following dominant part

$$N[G_A(\mathbf{r},\mathbf{r}_0)]\approx G_A\Big\{\frac{k^2}{2}\left[\varepsilon_{xx}(x-x_0)^2+\varepsilon_{yy}(y-y_0)^2+\varepsilon_{zz}(z-z_0)^2\right]- \\ -\frac{ikR_0}{16\varepsilon^{3/2}}(\nabla\varepsilon)^2\left(1-ikR_0\sqrt{\varepsilon}\right)\Big\}.$$

Consequently, as distinct from formerly inspected methods, the solution requires smoothness of medium heterogeneities changes, scilicet the smallness of first and second derivatives of the dielectric permittivity, but not of the dielectric permittivity itself.

In conclusion let's compare expressions for the Green's functions of observed approximations G_R, G_B and G_A. Factorization of exponential factor of function G_A in (7)

$$\sqrt{A(\mathbf{r},\mathbf{r}_0)} \approx 1+\frac{1}{2}\delta A(\mathbf{r},\mathbf{r}_0).$$

leads to expression, equal to formulas (10) – (11) for Rytov's approximation

$$G_R(\mathbf{r},\mathbf{r}_0) = G_0(R_0)e^{ikR_0\left[\frac{1}{2}\delta A(\mathbf{r},\mathbf{r}_0)-\frac{1}{24k^2}(1-ikR_0)\Delta\varepsilon(\mathbf{r})+\ldots\right]},$$

and factorization of exponential factor of G_R function into Taylor series up to its linear components carries into expression of Born approximation (9).

By this means the comparison of different approximate methods against each other demonstrates their remarkable similarity in attempts to specify Green's function phase dependence for heterogeneous medium. Also it's possible to notice how the smallest inaccuracies in determination of phase functional dependence leads to a significant drop in method's order of accuracy.

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