Study on Hermitian, Skew-Hermitian and Uunitary Matrices as a part of Normal Matrices

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Abstract— **A normal matrix plays an important role in the theory of matrices. It includes Hermitian matrices and enjoy several of the same properties as Hermitian matrices. Indeed, while we proved that Hermitian matrices are unitarily diagonalizable, we did not establish any converse; normal matrices are also unitarily diagonalizable. In this present paper we have tried to establish the proper relation of normal matrices with others.**

Keywords— **Matrices, Normal, Hermitian, Skew-Hermitian, Unitary, Diagonalzation.**

I. INTRODUCTION

Let A = (a_{ij}) be an $n\times n$ square matrix then matrix A is called symmetry if A = A^T and matrix A is called skew symmetry if A = - A^T when all the elements of the matrix are real. Let the elements of an $n\times n$ matrix A are complex except diagonal elements and $A=\left(\overline{A}\right)^T = A*$ then matrix A is said to be Hermitian matrix. It is called symmetric if it is Hermitian and real. The matrix A is called skew-Hermitian if $A=-\left(\overline{A}\right)^T = -A*$. A complex matrix A is called unitary if $A^{-1} = A*$ i. e. $AA^* = I$. The purpose of our paper is to study about the various results of Normal matrix and their relation with Hermitian, Skew-Hermitian and Unitary Matrices etc. The following are basic properties of Hermitian, Skew-Hermitian and Unitary Matrices:

(i). If a_{ii} is real then the elements on the leading diagonal of an hermitian matrix are real, because $\overline{a_{ii}} = a_{ii}$.

(ii) All the elements on the leading diagonal of a skew-Hermitian matrix are either purely imaginary or 0, this follows from the fact that $\overline{a_{ii}} = -a_{ii}$, so the real part of a_{ii} must equal its negative, and this is possible if a_{ii} is purely imaginary or 0.

(iii) Let the elements of an hermitian matrix are real, then the matrix is a real symmetric matrix, because $\overline{A}^T = A^T$, and the definition of hermitian matrix reduces to the definition of a real symmetric matrix.

(iv). Let the elements of a skew-hermitian matrix are real, then the matrix is a skew symmetric matrix, because then the definition of a skew-hermitian matrix reduces to the definition of a skew-symmetric matrix.

(v). Any n×n matrix A of the form A = B + iC, where B is a real symmetric matrix and C is a real skew-symmetric matrix, is an hermitian matrix. This follows directly from properties (iii) and (iv).

(vi). Any $n\times n$ square matrix A can be written in the form A = B + C, where B is hermitian and C is a skew-hermitian, then we can see that

$$A = \frac{1}{2}\left(A+\overline{A}^T\right)+\frac{1}{2}\left(A-\overline{A}^T\right) \quad , \quad B=\frac{1}{2}\left(A+\overline{A}^T\right)$$

and $C=\frac{1}{2}\left(A-\overline{A}^T\right)$, then it is easy to see that

$\overline{B}^T = \frac{1}{2}\overline{\left(A^T+\overline{A}\right)} = \frac{1}{2}\left(A+\overline{A}^T\right) = B$ and also we have $\overline{C}^T = \frac{1}{2}\overline{\left(A^T-\overline{A}\right)} = -\frac{1}{2}\left(A-\overline{A}^T\right) = -C$, this shows that matrix B is hermitian and C is skew-hermitian matrix.

(vii). A real unitary matrix is an orthogonal matrix, because in this case $\overline{A}^T = A^T$, causing the definition of a unitary matrix to reduce to the definition of an orthogonal matrix.

(viii). The determinant of a unitary matrix is ±1.

A square complex matrix A is diagonalizable if there exists a unitary matrix U with a diagonal matrix D such that $U*AU = D.$ The square matrix A is unitary diagonalizable if

$$A*A = AA*, \tag{1}$$

and if a matrix satisfying this property then it is said to be Normal matrix. Every hermitian matrix, every unitary matrix and every skew – hermitian matrix $(A* = -A)$ is Normal and if a square complex matrix is unitary diagonalizable it means that it must be normal.

II. DEFINITIONS, NOTATIONS AND RESULTS

Let α and β are complex numbers and A and B are two matrices with linearty property and if any linear combination αA + βB has an characteristic roots the numbers $\alpha\lambda_i + \beta\mu_i$ where λ_i and μ_i are the characteristic roots of A and B respectively both taken in a special ordering which is the generalization of the theorem given in [1]. Any square matrix with complex elements can be taken into a triangular matrix under a unitary transformation considered by [2]. If two normal matrices A and B holds property L then they commute, has been proved by [3]. Further he also proved that if a normal matrix has its characteristic roots in the main diagonal then it is diagonal matrix. The skew hermitian matrices can be characterized as the normal square roots or negative definite or semi definite, Hermitian matrices was studied by [4]. These matrices represents a set of generators of all like ranked square roots of such Hermitian matrices in the sense that every such square root is similar to a skew hermitian square root. Further the author [4] has proved the result as given below:

Every square Hermitian matrix is a normal square root of a negative definite, or semi definite, hermitian matrix, its

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converse is also true that every negative definite, or semi definite, hermitian matrix possesses matrix square roots then the normal matrices among which are skew hermitian. It is also true that every real skew matrix is a real normal square root of a negative definite or semi definite, real symmetric matrix, whose non-zero eigenvalues have even multiples and it is conversely true also that every negative definite, or semi definite, real symmetric matrix, whose non zero eigenvalues have even multiplicities, possesses real square roots, then the normal ones of which are real skew. The author of [4] has used the following lemma:

Lemma.

If A be a matrix with rank r is similar to a diagonal matrix then any k th root of matrix A is similar to a diagonal matrix. This lemma is a direct result of application of a method suggested by [5] for finding all k th roots of matrix or something directly by application of method provided by [6] to solve polynomial equations P(X) = A, with the help of this result he proved that if H be a hermitian negative , or semi-definite matrices of rank r, then every square root of rank r is similar to a skew hermitian square root of matrix H. For the square matrix which is defined over a field of characteristic 0 the equation

$$X\,Y - Y\,X = A \tag{2}$$

has solution X, Y if and only if Tr(A) = 0 has been studied by [7]. The above result was extended to the arbitrary field by [8]. We know that the square matrix A can be written as commutator (X Y – Y X) if and only if Tr(A) = 0. For a fixed field A the spectrum of one of the factors may be taken to be arbitrary while the spectrum of the other factor is arbitrarily as long as it has distinct characteristic roots was introduced by [9]. The author of [9] has proved the following theorem:

A. *Theorem*

Let $\lambda_1, \lambda_2, ..., \lambda_n, \lambda_{n+1}, ..., \lambda_{2n}$ be arbitrary complex numbers except that $\lambda_i \neq \lambda_j$ for $i \neq j$ and $i, j \leq j,$ then if Tr(A) = 0 there is a solution of X and Y to (2) with set of eigenvalue

$\sigma(X) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\sigma(Y) = \{\lambda_{n+1}, \lambda_{n+2}, ..., \lambda_{2n}\}.$

Further X may be taken to be normal matrix. For proof of the theorem he used the following lemma due to [10], [11]:

Lemma 1. If Tr(A) = 0, then matrix A is unitary equivalent to a matrix $B = (b_{ij})$ with $b_{ii} = 0, i = 1, 2, ..., n.$

Lemma 2. (Due to [12]) Let $a_{ij}, i \neq j,\ i, j = 1, 2, ..., n$ and $\alpha_1, \alpha_2, ..., \alpha_n$ be prescribed elements from an algebraically closed. If $A = (a_{ij})$ then $a_{ii}, i = 1, 2, ..., n$ may be chosen so that set of eigenvalues $\sigma(A) = \{\alpha_1, \alpha_2, ..., \alpha_n\}.$

An application of hermitian matrices to combinatorial optimization problems was given by [13]. If A is Hermitian and positive definite matrices, it is interest to find the Kantorovich ratio

$$\max_{ij} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \tag{3}$$

λ_i's are eigen values of a normal matrix $A = (a_{ij})_{n \times n}$.

The authors of [14] have been studied same inequalities relating the center and radius of smallest disc Γ containing these eigen values to the entries in normal matrix A. If applied to hermitian matrices the results of [14] gives the lower bounds on the spread $\max_{ij}(\lambda_i - \lambda_j)$ of A_i and if applied to positive definite hermitian matrices, this gives lower bounds on Kontorovich ratio (3). The quantity (3) governs the rate of convergence of certain iterative schemes for solving linear systems of equations AX = b [7, chapter 4]. In this situation we can easily show

$$\max_{i,j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \geq \max_{i,j} \frac{a_{ii} - a_{jj}}{a_{ii} + a_{jj}}$$

by using the fact the diagonal entries of A are convex combinations of the eigen values of A. The possibility of interest in matrix A and hermitian H for which two results

$$AH + HA^* = I \quad \text{and} \quad HA + A^*H = I \tag{4}$$

was studied by [15]. Further author of [16] relates certain cases of it to the normality of matrix A. It is inciting to hypothesis that (4) has a solution iff matrix A is normal. The author of [16] has obtained the various criteria for normality of A in terms of hermitian solutions of the equation which satisfy additional conditions. He proved the interesting result that if $\ln A = (\pi, \gamma, 0)$ where triple (π, γ, δ) be the inertia, π be the number of eigen values with positive real part, γ be the number with negative real part and δ be the number with zero real part, then A is normal iff there is a hermitian matrix H for which both $AH + HA^* = I$ and AH – HA = 0, while the authors of [17] have been proved $\ln H = \ln A = (\pi, \gamma, 0)$ from main inertia theorem.

Let A be a square complex matrix and a hermitian solution G is sought the equation

$$AG + GA^* = A \tag{5}$$

was studied by [15, 18]. A necessary and sufficient condition for equation (4) was established by [19] for the existence of a hermitian solution H. The study of equation (4) was initiated by [16], where he has shown that matrix A is stable if and only if A is normal.

Let A be a n×n normal matrix then (1-70) conditions are equivalent to (1) each of which is equivalent to normal matrix A was studied by [20]. The condition of normality is a strong one but as it admits the hermitian unitary and skew-hermitian matrices, it is very important one which often appears as the appropriate level of generality in high algebraic work and for numerical results which deals with perturbation analysis. At the end of the introduction the authors of [20] say: “Reflecting the fact that the normality arises in many ways, it hoped that not only will it be useful now, but its utility will grow over time as conditions added”. Nearly after a decade authors of [21] have been added more twenty conditions that conditions (71-90). The author [22] has presented the matrix $A \in C^{N \times N}$ is normed if and only if for all vectors $x \in C^N$

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$$\left\| A^{n+m} \right\|_2 \le \sqrt{\left\| A^{2n} x \right\|_2 \left\| A^{2m} \right\|_2},$$

for all $n, m = 0, 1, ...$ where $\| \ \|_2$ be the Euclidean norm on C^N. The Lexicographic order is a total in C compatible with addition of complex numbers and multiplication by positive real and it is characterized by its positive cone

$$H = \{\alpha + i\beta : \alpha > 0 \text{ or if } \alpha = 0, \beta > 0\}.$$

The compatibility with addition is $H + H \subseteq H$ which compatibility with multiplication by positive real, is $\lambda H \subseteq H$ for $\lambda > 0$. The order being total is $H \cup -H = C \setminus \{0\}$. The lexicographic order is not Archimedian and apart from rotation if the positive cone is the only total order in C compatible with these addition and multiplication operations. The difference between hermitian and general normal matrices is that they can have as eigen values arbitrary complex number C of course is not an ordered field, but it turns out the simple fact that C can be totally ordered as a vector space over the reals if enough to find useful information on spectra of normal matrices by using hermitian matrices as an inspiration was given by [23].

H-Unitary Matrix

A complex matrix that are unitary with respect to indefinite inner product induced by an invertible hermitian matrix H is said to be H-unitary matrix.

Lorentz Matrix

The real matrix that are orthogonal with respect to indefinite inner product induced by an invertible real symmetric matrices are said to be Lorentz matrices.

Let $M_n = M_n(F)$ be the algebra of $n \times n$ square matrices with entries in the field F = C, the complex numbers, or F = R the real numbers, and if $H \in M_n$ is an invertible hermitian matrix, a matrix $A \times M_n$ is said to be H-unitary if $H^* HA = H$. The authors of [24] and [25] have been presented applications of H-unitary valued functions in engineering and interpolation and for an exposition from the point of view of numerical method were studied by [26]. Several canonical forms of H-unitary matrices and demonstrate some of its applications was established by [27].

Conjugate Normal Matrices

Let $M_n(C)$ be the set of complex $n \times n$ matrices and a matrix $A \in M_n(C)$ is called conjugate normal matrix if

$$AA^* = \overline{A^* A}. \tag{6}$$

It plays an important role in the theory of unitary congruences as conventional normal matrices do in the theory of unitary similarities. We can easily verify that matrix $A \in M_n(C)$ is conjugate normal if and only if the corresponding matrix $\hat{A}$ is normal in the conventional sense, where

$$\hat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}.$$

One of the most useful criteria that $A \in M_n(C)$ is normal if and only if the hermitian adjoint A^* can be represented as a polynomial of A as $A^* = f(A)$.

Let the spectrum of A is $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, then desired polynomial f can be obtained by Lagranges interpolation

$$f(\lambda_i) = \overline{\lambda_i}, i = 1, 2, ..., n-1.$$

The degree of polynomial is at most n-1, and it coefficients are in general complex. The author of [28] used this criterion to show the following result:

Result

A matrix $A \in M_n(C)$ is conjugate normal if and only if the transpose A^T can be represented in the form

$$A^T = g(A_k)A$$

where g is a polynomial with real coefficients.

Condiagonalization

A matrix $A \in M_n(C)$ is called condiagonalizable if $A_R = A\overline{A}$ or $A_L = \overline{A}A$ is diagonalizable by a similar transformation or we can say that matrix $A \in M_n(C)$ is condiagonalizable if there exists a non-singular $S \in M_n(C)$ such that $S^{-1} A \overline{S}$ is diagonal.

The author of [29] has given a description of condiagonalizable matrices that would be more elementary then the use of the Canonical Jordan like form. He proved that any condiagonalizable matrix can be brought by a consimilar transformation to a special block diagonal form with the diagonal blocks of order 1 or 2.

Let λ be a simple eigen value of a normal matrix A, then its condition number attains the minimal possible value 1. In most general case where matrix A have multiple eigen values, a suitable characterization of ideal condition can be obtained from the Bauer-Fike theorem as below:

B.*Bauer-Fike Theorem*

Let $M_n(C)$ be the set of nxn complex matrices and a matrix $A \in M_n(C)$ be a diagonalizable matrix with eigen value decomposition

$$A = P \wedge P^{-1} \tag{7}$$

Let a matrix $B \in M_n(C)$ be an arbitrary matrix regarded as a perturbation of A, then for every eigen value μ of B, there exists a eigen value λ of A such that

$$|\mu - \lambda| \le cond_2 P \|B - A\|_2$$

where $\| \ \|_2$ be the 2-norm of the corresponding matrix and $cond_2 P = \|P\|_2 \|P^{-1}\|_2$ is the 2-norm or spectral condition number of P. For a normal matrix A the eigen vector matrix P in formula (7) can be chosen to be unitary, then $cond_2 P = 1$ and the author of [30, p-54], has proved the following proposition:

Proposition

Let $A \in M_n(C)$ be a normal matrix and $B \in M_n(C)$ be a perturbation of A, then for every eigen value μ of B, there exists an eigen value λ of A such that

$$|\mu - \lambda| \le \|B - A\|_2 .$$

The authors of [31] have been proved that complex symmetric matrices and more generally the entire class of conjugate normal matrices may be equipped with scalar characteristics that unlike eigen values are very stable to matrix perturbation. The class of normal matrices is important throughout the matrix analysis was given by [32]. This is especially important in matters related to similarity transformations and even more especially to unitary similarity transformations. A survey of the properties of conjugate normal matrices was presented by [33] they have also presented a list of conjugate normal matrix (6).

Normal Toeplitz Matrix

An infinite Toeplitz matrix is normal if and only if it is a rotation and translation of a Hermitian Toeplitz matrix.
Hermitian Toeplitz matrices play an important role in the trigonometric moment problem, Szeg¨o theory, stochastic filtering, signal processing, biological information processing and other engineering problems. A matrix $A \in C^{n \times n}$ is said to be centrohermitian [10], if $JAJ = \overline{A}$, where $\overline{A}$ be the element-wise conjugate of the matrix and J is the exchange matrix with ones on the cross diagonal means lower left to upper right and zeros elsewhere. Hermitian Toeplitz matrices are an important subclass of centrohermitian matrices and have the following form:

$$H = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ \overline{h_1} & h_0 & \cdots & .. \\ .. & .. & \cdots & .. \\ \overline{h_{n-1}} & .. & \overline{h_1} & h_0 \end{bmatrix}$$

A vector $x \in C^n$ is said to be hermitian if $Jx = \overline{x}$. Let $A \in C^{n \times n}$ be a hermitian centrohermitian matrix and $x \in C^n$ be an eigenvector of A associated with the eigenvalue λ, then $Ax = \lambda x$ implies $AJ\overline{x} = \lambda J\overline{x}$, which means that $x + J\overline{x}$ is also an eigenvector of A associated with the eigenvalue λ, and $x + J\overline{x}$ is hermitian. So we claim that an hermitian centrohermitian matrix A has an orthonormal basis consisting of n hermitian eigenvectors. Naturally, an hermitian Toeplitz matrix also has an orthonormal basis consisting of n hermitian eigenvectors.

Hankel Matrix

The normal Hankel problem is the one of characterizing the matrices that are normal and Hankel at the same time. Let $S = \{a_0 = 1, a_1, a_2, ...\}$ be a sequence of real numbers, the Hankel matrix is generated by is the infinite matrix is below,

$$H = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\ a_3 & a_4 & a_5 & a_6 & a_7 & \cdots \\ .. & .. & .. & .. & . & \cdots \end{bmatrix}$$

Hankel matrices are formed when, given a sequence of data, a realization of an underlying state-space or hidden Markov model is desired. The singular value decomposition of Hankel matrix provides a means of computing the A, B, and C matrices which define the state-space realization. The Hankel matrix formed from the signal has been found useful for decomposition of non-stationary signals.
The authors of [34-36] was posed and solved the problem of characterising normal Toeplitz matrices, they have also presented the interesting algebraic literature. The different of solution of this problem have been proposed by several authors. The author of [28] was posed the problem of characterizing normal Hankel matrices in 1997, and he has proved to be much difficult in comparision to Toeplitz problem. Suppose NH_n be a set of normal Hankel matrices of order n, and a certain specific subsets of this set were given in [37] also he has added an additional subset in 2007. The authors of [39] were able to extract few specific types of normal Hankel matrices from these conditions to class 2 and class 3 and they indicated the scalar multiples of unitary Hankel circulants. The authors of [38] have also proposed a new approach to solve the normal Hankel problem in 2007. By this approach the known subsets of NH_n are particular cases of a unified scheme. The authors of [41] have obtained a complete solution of the normal Hankel matrix problem. They have given the solution by the list of ten sub-classes of normal Hankel matrices NH_n .

The authors of [42] have proposed a general approach for computing the eigen values of a normal matrix, exploiting there by the normal complex symmetric structure. They have also presented an analysis of the computational cost and numerical experiments with respect to the accuracy of the approach. Further the authors of [42] have investigated the case of non simple singular values and propose theoretical frame work for retrieving the eigen values, and they also highlighted some numerical difficulties inherent to this approach. They have presents simple case where the intermediate matrix is symmetric, showed overall, good numerical performance both in terms of speed as well as accuracy, also presented several new directions for extending the presented research.

In numerical linear algebra algorithms for computing eigen values and singular values of matrices are amongst the most important ones. The authors of [43, 44] have been provided an incredible range of various methods iterative (ex Lanczos, Arnoldi) as well as the so called Direct methods viz. Divide and Conquer Algorothms, GR-methods. Many of the procedures are based on QR-method for computing eigen values and / or singular values. The QR-method consists of two steps:

A preprocessing step to transform the matrix A to a suitable shape admitting low cost iterations in the second step, this first step is essential since generically it reduces the global computational complexity of the next step with one order (ex. from $o(n^4)$ to $o(n^3)$). The definition of suitable shape depends heavily on the matrix type used. The second step consists of repeatedly applying QR-steps on the matrix until the eigen values are revealed. A constructive procedure to perform a unitary similarity transformation of a normal matrix with distinct singular values, to complex symmetric form was studied by [45]. In [45] the presented algorithm is capable of performing the transformation in a finite number of floting point operations. Further, he has discussed the possibility and presented a new method for computing eigen values of some normal matrices based on this transformation, he has also given the reduction as well as some of its properties. Another solution to this problem was given by the author of [28] in 1993.

The case when the Hermitian part $H(A) = \frac{1}{2}\left(A + A^*\right)$ of a complex matrix $A \in C^{n\times n}$, with the same rank as A, its idempotent is motivated by an application to statistics related to Chi-square distribution was introduced by the author of [46]. This result was extended by [47] by relaxing the assumption on the rank. The generalization of these results concerning H(A) as well as study the corresponding problem for the skew-Hermitian part $S(A) = \frac{1}{2}\left(A - A^*\right)$ of A was studied by [48]. The result related to the products of singular symmetric matrices was given by [49] as follows:

C. *Theorem*

Let A and B be real n×n symmetric matrices with eigen values $\lambda_1, \lambda_2, ..., \lambda_r, 0, ..., 0$ and $0, 0, ..., 0, \lambda_{r+1}, ..., \lambda_n$ $(\lambda_i \neq 0, 1 \le i \le n)$, respectively. If A + B has eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ then AB = 0.

The above theorem was soon generalized by [50] to normal matrices. The most remarkable property of the normal matrices is that they are unitary diagonalizable. There are two difficulties that a sum of normal matrices need not be a normal and principle sub matrices of a normal matrix need not to be normal. These two obstacles was bypassed by Djokovic successfully and he extended above theorem as follows:

D. *Theorem*

Let N_i, $1 \le i \le k$ be n×n normal matrices and N_i has non-zero eigen values $\lambda_1^{(i)}, \lambda_2^{(i)}, ..., \lambda_{r_i}^{(i)}, 1 \le i \le k$ and $r_1 + r_2 + ... + r_k \le n.$ If $N := \sum_{i=1}^{k} N_i$ has non-zero eigen values $\lambda_j^{(i)}, 1 \le j \le r_i, 1 \le i \le k$, then N is a normal and $N_i N_j = 0$ for $i \ne j.$

The author of [51] has pointed out several implications of result of [50] concerning orthogonality of normal matrices which satisfy a certain condition on the eigen values of their sum. Further he has proved an analogous result in the setting of conjugate normal matrices.

III. CONCLUSION

In the theory of matrices, normal matrices and its properties beers very large range of new results. The present subject matter related to the study of normal matrices is not very exhaustive. It is known that the normal matrices are perfectly conditioned with respect to the problem of finding their eigen values. We have tried to correlate and present the variety of problems of normal matrices.

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