Stochastic Models of Traffic Management

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***Abstract*— This article discusses the problem of blocking nodes in road networks and percolation thresholds for the transport infrastructure of a modern metropolis. For metropolitan networks, the values of percolation thresholds are calculated and displayed, considering the different density of connections between network nodes. Further, it is shown that the dependence of the values of the percolation thresholds on the network's density can be described by functional dependencies with a high degree of correlation.**

The obtained results can be used to assess the reliability of transport infrastructure and to check the increase in the capacity of selected sections of the road network.

Further, the article discusses obtaining a description of road infrastructure from open sources (obtaining data from OpenStreetMap using the SUMO - Simulation of Urban MObility package), after which it is possible to build a graph of the road network and determine its percolation properties. For the constructed nodes of the graph described a model of the stochastic dynamics of blocking a single road lane in the transport network.

The threshold value L of number of cars that can be placed in the lane (based on the length of the road lane) is used as constraints and the incoming and outgoing flows of cars are also determined as income parameters of a model. The constructed model allows us to obtain the predicted blocking time of a road network line, for a given probability of such blocking, where the probability of blocking a single road network line is taken from the percolation properties of this network discussed earlier.

The resulting blocking times of road network nodes make it possible to build an algorithm for controlling traffic light regulation.

***Keywords*—percolation theory, road networks, roads modelling.**

I. Introduction

The management of a city's road network is an important task, as it contributes to economic growth and the development of the urban environment. In today's world, increasing attention is being paid to ecological issues, which requires improving the efficiency of transport and transport management. This includes effective planning and managing of road infrastructure that helps regulate traffic, reduce congestion, and reduce travel times. In most cases, there is a shortage of roads space, leading to congestion on roads and, consequently, loss of time, economic problems, and increased emissions of harmful substances.

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II. Relevance of the Study

The modern transport network is a complex dynamic system, with many road users (cars and trucks, personal transport such as scooters and bicycles), with many components of the road network (roads, dedicated roads, toll roads, roads with time-varying characteristics, complex road junctions, dynamically controlled traffic lights and ordinary (uncontrolled) traffic lights, barriers) and with the perspective of appearing on city streets completely Autonomous transport [1], that kind of systems are becoming more complex from year to year.

Currently, many scientific works are devoted to the problems of urban transport networks, route optimization and traffic management. Some of the most common topics of modern research are presented in fig. 1. Researchers develop and apply tools to simulate congestion and transport networks and to predict the behavior of road users to improve the quality and quantity of road networks.

Fig. 1 shows the main directions of modern research, such as:

- Descriptions of the cities of tomorrow.
- Construction of intelligent transport systems.
- Application of multi-agent systems for traffic management.
- Using different genetic algorithms to optimize the operation of the urban transport network.
- Construction of optimal algorithms for the selection of a route and the conduct of linear planning with the assignment of a route.
- Application of various recurrent neural networks to predict urban traffic flows.

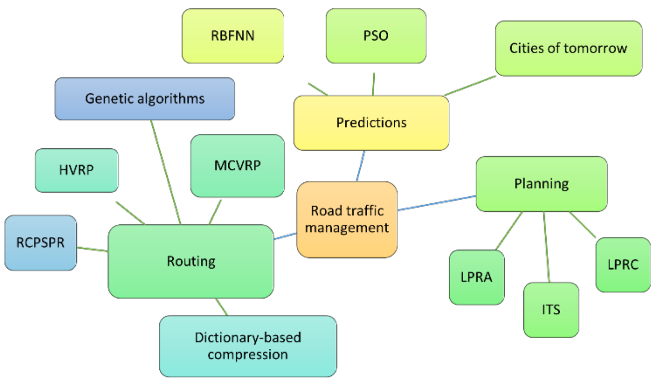

Fig. 1. Basic approaches to transport network modeling

In addition to all the efforts made, we notice that with the growth of cities, with the advent of the concept of a smart city, the city of tomorrow and with the expansion of road networks – the logistical tasks in the urban environment are also becoming more complicated. Despite the abundance of proposed solutions on the roads, congestion is still observed, which leads to significant economic (man-hours), environmental (excess emissions into the atmosphere) and many other problems (inefficient use of transport infrastructure, for example).

III. Problem Statement

Significant indicators of the density of motor vehicles per unit area of all available roads lead to the inevitable emergence of large accumulations of cars on certain elements of the road transport network (intersections and roads), i.e., the occurrence of congestion (traffic jams). The bulk of all studies of transport traffic, its analysis and development of management models are aimed at solving the problem at the local level, while not affecting the transport infrastructure of the city.

The road and transport network of modern megacities has a very large, complex, and extensive structure (see, for example, fig. 2. When modeling traffic, it is necessary to consider the dynamics of changes in traffic congestion (daily change in the intensity of flows) and the fact that all elements of the transport network graph (nodes and edges) have different characteristics (capacity).

Fig. 2. Fragment of modern transport infrastructure (SUMO)

If we go the way of creating a detailed model of the transport network graph with a detailed description of the attributes of all elements (the number of lanes on the roads, their length, the number of directions at intersections, etc.), which is required for local management of intersections, then such a model will be extremely complex and difficult to implement for practical application. From the point of view of labor intensity and applicability, it is more acceptable to create a percolation model of the city's transport network. In such a model it is possible to ensure the overall operability of the structure, even though individual elements may be blocked due to the formation of traffic jams. In this case, by ensuring operability and availability, we will understand that between any two arbitrary nodes of the network, there is at least one free path from unblocked elements of the road network.

IV. Percolation in Road Networks

In the percolation theory for networks with different structures (regular, random), the solution of nodes problem and the links problem are studied in previous papers [2–5]. When we are solving the links problem, determine the proportion of links that need to be broken so that the network splits into at least two unrelated parts (or vice versa, the share of conductive links when conduction occurs). In the node problem, the proportion of blocked nodes is determined, in which case the network will break up into unrelated clusters within which links are preserved (or vice versa, the proportion of conductive nodes when conduction occurs). The proportion of unblocked nodes (in the node problem solving task) or unbroken links (in the links problem solving task) in which conduction occurs between two arbitrarily selected network nodes is called the percolation (flow) threshold. For the same structure, the values of the percolation thresholds for the links problem and the node task have different values. Note that, when a node is blocked, all its links are blocked, and when links are blocked, only one connection between neighboring nodes is blocked.

The using of the concept of blocked node shares or blocked links shares is equivalent to the concept of the probability of finding a randomly selected node (or link) behind a blocked state. Therefore, it can be assumed that the value of the percolation threshold determines the probability of passing through the road network, even though some part of its nodes (or links) can be blocked (excluded), i.e., the average probability of blocking a single node (or link) is set.

Reaching the percolation threshold in the network corresponds to a cluster (for example, blocked streets) in which there are connections between any of its arbitrary nodes. At this point is formed cluster, also-called infinite or contracting.

For finite-size structures, conduction may occur with different proportions of conductive nodes (or links, see fig. 3). However, if the size of the network L is going to infinity, the area of transition to the conductive state becomes compact (see fig. 3, where curve I is represented for a structure of small size, and II for an infinite network). Note, that this approach claims to be universal and can be applied not only to the topology of road networks, but also to other topologies [6, 7].

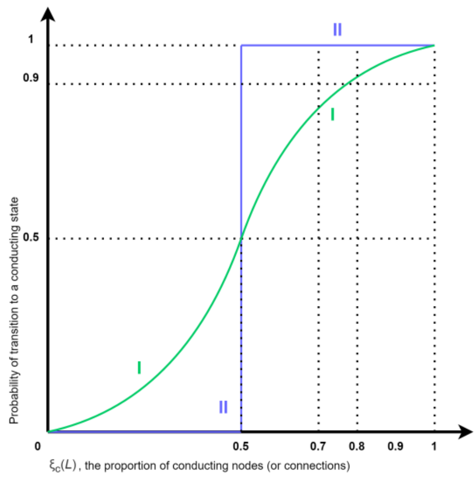

Fig. 3. The probability of percolation depending on the size of the proportion of conductive nodes (or links).

For finite-size structures, the value of the percolation threshold $\xi_c(L)$ can be determined from a given value of the probability of the network entering a conductive state. On fig. 3 such kind of probability is selected as 0.5 (50%). However, you can take, for example, a value of 0.90, 0.95 or 0.99 (then the percolation threshold will meet the specified criterion for network reliability), i.e., it is possible to determine at what proportion of blocked nodes and/or links the entire network will lose the desired level of operability.

If you specify a network availability value (the probability of a transition or being in a conductive state), you can determine the proportion of nodes (or links) that are in an unblocked state.

The proportion of blocked nodes (or links) in which the conductivity of the network disappears (which can be calculated by the formula: one minus the proportion of conducting nodes (or links)), sets the blocking of the entire network, and the value of which can be associated with the macro characteristics of the traffic in the transport system (at current point in time). In the simplest case, we can give the following estimate: the accepted level of intensity of free traffic (let's denote it as q_{max}): for European cities is 600-900, in the USA - up to 1300, and in Russian cities this figure is 300-700 cars per lane per hour. Accordingly, knowing the total length of the roads of the metropolis and the number of lanes on them, as well as the dynamics of the daily number of vehicles, it is possible to calculate based on these data the average level of traffic intensity for the entire network at any time (let's denote it as $q(t)$). Then, the average probability ($P(t)$) of blocking the network element at time t can be determined as follows:

$$P(t) = \begin{cases} \frac{q(t)}{q_{max}}, when\ q(t) \leq q_{max} \\ 1, when\ \ q(t) > q_{max} \end{cases} \qquad (1)$$

Further, using the obtained estimate of the probability of blocking the network element, it is possible to determine at a given time t the state of its reliability and operability, as well as to analyze the daily dynamics of changes, and, if necessary, change the structure (for example, the density of links) of the transport system in the necessary way.

For accurate estimates of the average probability of blocking, various macroscopic mathematical models of traffic flows can be used (models of Greenshields, Richards, Grindberg, El Khozaini, Underwood, Drake, Pipes, optimal speed, "smart" driver, following the leader, cellular automata, and many others).

The main problem in the study of percolation properties of network structures that have a random structure is the absence (at present) of general analytical methods, and the study of network structures is possible mainly only with the involvement of computer modeling methods. To study the percolation properties of planar network structures that have a random structure, it is first necessary to build their topological graph, which is a very time-consuming task. At the same time, to build a graph, you can use the algorithm for building planar networks [6,10]. After the network is built (either algorithmically or based on real data), it is possible to study the percolation properties of the resulting network structure.

The application of some methods of percolation theory to the modeling of traffic flows is described in the work [7]. In this paper, traffic dynamics is considered as a critical phenomenon in which there is a transition between isolated local and global flows on the roads with the formation of clusters of congested sections of the transport network in local structures and their unification into a giant cluster. Local flows are connected by narrow links, and the narrow links themselves can occur in different places of the transport network at different times of the day. The authors [7] characterize such processes as traffic percolation between local clusters. The authors tried to describe how local flows on roads interact and unite into a global flow throughout the urban network. One of the complex tasks that arise when modeling a transport structure is the assessment of all its dynamics and traffic organizations throughout the network and its relationship with local traffic characteristics [7]. They collected and analyzed the speeds of more than 1,000 roads with a record of 5-minute segments measured on roads in a central area of Beijing. The data covers a period of 2 weeks in 2013, the road network consists of intersections (nodes) and road sections between two intersections. For each road, the speed of $V_{ij}(t)$ changes during the day according to the real time. For each *road* e_{ij} , the authors set the 95th percentile of its maximum speed in each day and defined the model parameter $r_{ij}(t)$ as the ratio between the current speed and the limited maximum speed measured for that day. At some given threshold q, all roads e_{ij} can be divided into two categories: those that are accessible at $r_{ij} > q$ and inaccessible at $r_{ij} < q$. According to the authors [7], in this way, it is possible to construct a functional traffic network for a given value q from the dynamics of traffic flows.

With $q = 0$, nothing happens to the network with traffic, and with $q = 1$, the traffic becomes completely fragmented. Hierarchical traffic organization at different scales appears only in groups of roads with r_{ij} higher *than* q. Such clusters are functional modules consisting of connected roads at speeds above q. For example, at $q = 0.69$, a speed occurs on some sections that cannot be maintained by a transport network everywhere. When the value of q is reduced to 0.19, then, small clusters merge together and a giant cluster is formed in which a functional network (with a lower flow rate) extends to almost the entire road network.

The advantage of the method of traffic modeling and analysis presented by the authors is that having data on transport flows in a real network, it is possible to determine the critical value q_c below which the transport network loses functionality (percolation threshold). In [7], the value of q_c was approximately equal to 0.4.

The disadvantage of the study [7] is that the results obtained are of a private nature and are obtained only for a certain part of Beijing's transport network. Therefore, they cannot be generalized to a transport network with an arbitrary structure.

In addition, another drawback is the significant laboriousness of the method of analysis and modeling of transport networks proposed by the authors of this work.

A more technological and universal modeling method can be the use of general characteristics of network structures, for example, the influence of network density on traffic percolation. In this case, if it turns out that the density of the network, regardless of its actual structure, is a universal characteristic that allows you to link structural and dynamic (such as traffic) characteristics, then, at a minimum, this reduces the complexity of analyzing and modeling the performance of the transport network, and this approach will be a universal solution.

V. Finding solutions

As part of the study, random planar networks, and algorithms for modeling percolation processes in them, were simulated.

The results of numerical modeling based on algorithms and the calculation of percolation thresholds of planar networks with a random number of links on each node for the nodes problem task and links problems task are presented in Table 1 below.

TABLE I. Values of percolation thresholds for planar networks having a random structure

№	*Task Type*	*Density (Inverse density)*[a]	*Percolation threshold (ln P(t))* [b]
1.	Nodes problem task [9–12]	5.99 (0.167)	0.500 (-0.693)
2.		5.40 (0.185)	0.533 (-0.629)
3.		4.80 (0.208)	0.570 (-0.562)
4.		4.50 (0.222)	0.593 (-0.523)
5.		4.20 (0.238)	0.618 (-0.481)
6.		3.90 (0.256)	0.650 (-0.431)
7.		3.60 (0.278)	0.683 (-0.381)
8.		3.42 (0.292)	0.708 (-0.345)
9.		3.18 (0.314)	0.750 (-0.288)
10.		2.94 (0.340)	0.793 (-0.232)
11.		2.70 (0.370)	0.852 (-0.160)
12.		2.46 (0.407)	0.925 (-0.078)
13.	Links problem task	5.99 (0.167)	0.395 (-0.929)
14.		5.69 (0.176)	0.405 (-0.904)
15.		5.39 (0.186)	0.435 (-0.832)
16.		5.09 (0.196)	0.445 (-0.810)
17.		4.49 (0.223)	0.480 (-0.734)
18.		4.19 (0.239)	0.510 (-0.673)
19.		3.89 (0.257)	0.550 (-0.598)
20.		3.59 (0.279)	0.570 (-0.562)
21.		3.29 (0.304)	0.625 (-0.470)
22.		2.99 (0.334)	0.685 (-0.378)
23.		2.87 (0.348)	0.715 (-0.335)
24.		2.70 (0.370)	0.770 (-0.261)
25.		2.58 (0.388)	0.805 (-0.217)
26.		2.39 (0.418)	0.900 (-0.105)

[a.] Average number of links per network node (network density). In parentheses are the values for inverse density values.

[b.] Percolation threshold value (the proportion of conductive nodes or links at which conductivity of the network occurs). In brackets are the values of the natural logarithm values from the value of the percolation threshold.

Table 1 specifies the values of percolation thresholds, as the proportion of conductive nodes (or links) at which the conductivity of the network occurs. The proportion of blocked nodes (or links) in which the conductivity of the network disappears can be found as a one minus the proportion of conductive nodes (or links).

Note that the values of percolation thresholds of planar networks with different densities for the problem of blocking nodes were determined earlier in the works [9–12], where, for numerical modeling, networks consisting of 100,000 nodes were used. To conduct numerical experiments in solving the links problem, smaller networks with 5,000 nodes were used for this work, which was due to the need to use insignificant computing power to solve problems.

As a level for determining the threshold of percolation of network structures, the value of the probability of the network transition to a conductive state of 0 is selected. Fig. 5 (see fig. 3). However, once again, we note that you can take another value of the transition probability , for example, 0. 90, 0.95 or 0. 99 (then the percolation threshold will specify the criterion of network reliability), i.e. it is possible to determine under what proportion of blocked nodes and / or links the network as a whole will lose the desired level of operability, or in another interpretation: how much it is necessary to maintain the level of availability of the transport network.

It is important to note that for a flat graph, the average number of links per node (network density) cannot exceed the value of 6. This is a consequence of Euler's theorem [13], according to which the equality must be fulfilled for a connected planar graph: $V–E+F=2$, where V is the number of vertices in the graph, E is the number of edges, F is the number of regions into which the graph divides the plane.

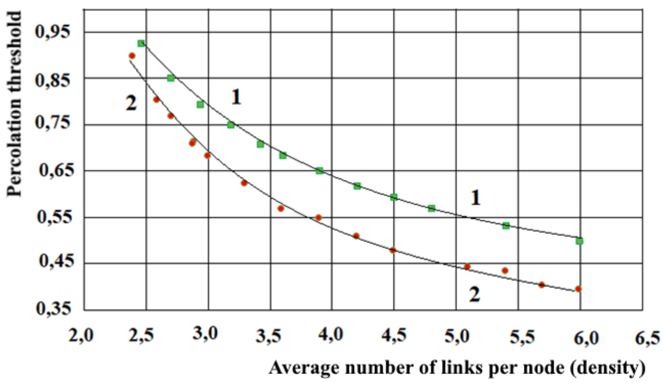

Fig. 4. Dependence of the values of the percolation thresholds of planar random networks on their density (curve 1 - for the nodes problem task, curve 2 - for the links problem task)

Fig. 4 shows the dependencies of the values of the percolation thresholds of planar networks on the average number of their links per node (in the node blocking problem [8] and in the link blocking problem).fig. 4 [9]

To determine the effect of the density of network structures on the value of their percolation thresholds, it is necessary to analyze the data presented in table I and fig. 4 and obtain a functional relationship that can describe the effect of network density on the value of the percolation threshold of this network. This will allow, having determined the density of connections of real transport networks, to estimate the value of their percolation threshold and, thereby, to draw a conclusion about the reliability of their structure, i.e., to determine at what proportion of blocked nodes and / or connections the network, will lose the desired level of operability.

The results obtained can be used in the construction of transport networks or the reconstruction of such networks to increase capacity and operability, as well as to obtain estimates for improving the reliability of transport infrastructure before the actual construction phase is completed.

In the works [14–16], based on the ideas of Shklovsky and de Genna about the topological structure of the connecting cluster ("skeleton" and "dead" ends), a function of the conditional probability of flow (percolation) in the lattice Y(ξ,L) was obtained, which has the form (the probability of reaching the percolation threshold with a certain proportion of blocked nodes is considered):

$$Y(\xi, L) = \frac{1}{1+e^{-S(\xi,L)}} \qquad (1)$$

where $S(\xi, L) = \sum_i a_i(\xi^i - \xi_c^i(L))$ is a polynomial of degree i, a_i is its coefficients, ξ is the proportion of blocked nodes, $\xi_c(L)$ is the proportion of blocked nodes corresponding to the value of the percolation threshold, and it depends on the network size L. A polynomial $S(\xi, L)$ of degree i can depend on the topological properties of the network structure (network density, spatial symmetry, dimension, etc.), which can be specified in the phenomenological approach of description using coefficients a_i.

The main problem in describing percolation using equation (1) is to determine the degree of the polynomial i and its coefficients. The combined use of the function (1), as well as the methods of Hodge's algebraic geometry [18] and the Kadanov–Wilson similarity theory [18, 19] using renorm groups [5], makes it possible (in some cases) to calculate the theoretical values of the percolation threshold for some regular structures [13–16]. Hodge's theory deals with algebraic manifolds (sets composed of subsets, each of which is a set of solutions to some polynomial equations), and the geometric representation of algebraic manifolds are figures called Hodge cycles. Linear combinations of such geometric shapes are called algebraic cycles [20].

Let us propose another approach, which consists in the fact that it is possible to express the dependence of a polynomial $S(\xi, L)$ of degree i on the conditional probability of $Y(\xi,L)$ of the flow in the lattice and determine the influence of topological factors on this dependence. Using the formula (1) you can get:

$$ln\, Y\,(\xi, L) = -1\{1 + e^{-S(\xi,L)}\} \qquad (2)$$

where $S(\xi, L) = \sum_i a_i(\xi^i - \xi_c^i(L))$is a polynomial of degree i, and a_i is its coefficients, ξ is the current value of the fraction of blocked nodes, $\xi_c(L)$ is the proportion of blocked nodes corresponding to the value of the percolation threshold (depends on the network size L). Given that near the percolation threshold $\xi \approx \xi_c(L)$, we get that the magnitude of the polynomial S(ξ,L)$S(\xi, L)$ is small and can be decomposed$e^{-S(\xi,L)}e^{-S(\xi,L)}$ into a series, limiting itself to two of its members. Having done the transformations, we get:

$$ln\, Y\,(\xi, L) \approx 1 - S(\xi, L) = 1 - \sum_i a_i(\xi^i - \xi_c^i(L)) \qquad (2)$$

The right side of equation (2) may be a function (or functional) of several variables, each of which is related to a specific topological property of the network. For example, one of the parameters may be the average number of connections (network density).

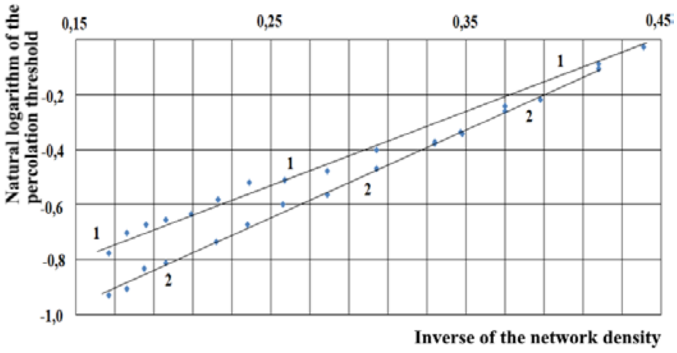

Fig. 5. . Dependence of the natural logarithm of the percolation threshold (ln$P(x)$) of planar random structures on the inverse of their density ($1/x$)

The described approach makes it possible to analyze the data given in 0 and fig. 3 and present them as a dependence of natural logarithms of the values of the ln $P(x)$ percolation thresholds from topological characteristics (for example, the inverse of the network density, ($1/x$) is a unit divided by the average number of connections per network node (see fig. 5).

As can be seen from fig. 5, the resulting dependencies have a linear form and can be approximated by linear equations.

For planar structures in the nodes problem task, the dependence of the natural logarithm of the percolation threshold ln$P(x)$ on the inverse of the network density ($1/x$) can be described by the equation:

$$ln\, P_{node,unreg}\,(x) = \frac{2{,}52}{x} - 1{,}08 \qquad (3)$$

where the correlation coefficient of the numerical data and the linear dependency equation is equal 0. 99 (see straight 1 on fig. 5). Similarly, in the links problem task:

$$ln\, P_{node,unreg}\,(x) = \frac{3{,}19}{x} - 1{,}44 \qquad (4)$$

with the value of the correlation coefficient of numerical data and the equation of linear dependence equal to 0.99 (see straight 2 on fig. 5).

VI. Stochastic model of a road section

Fig. 6 shows a fragment of the road network built from OpenStreetMap data using the SUMO (Simulation of Urban Mobility) package. Such of data illustrates a conditional regulated intersection in a real city. Let's consider the composition and characteristics of this section of the road network in more detail way.

First, we note that the intersection itself cannot be a "node" of the road graph, since according to the rules of the road, the car should not stop at the intersection and cannot go to it if there is a traffic jam ahead [21]. Given this limitation, the "intersection" does not have its own state, with the vehicle on it, and, thus, is a union of various "connections" (movements through the intersection) of the graph of the road network.

Secondly, when we choosing one lane as a "node" of the road network (with its beginning, end and possible directions of movement), we must take into account the physical parameters of this lane, such as the length of the lane, based on which depends the number of cars that can occupy space on this lane, or the capacity of this lane, as well as the connections of this lane with other lanes: first of all, connections passing through regulated and unregulated intersections, but also connections with neighboring lanes - "on" which and "with" which can be cars move into maneuver.

Thus, for our graph of the road network (consisting of separate road lanes), we want to understand the indicators of traffic reliability on these lanes (find the probability that the number of cars will not exceed the maximum threshold) and build work on managing traffic flows in this network from such indicators.

Special attention in Fig. 6 deserves the possible directions of movement through the intersection (indicated by white arrows), which may differ, and which can be regulated by individual traffic light phases.

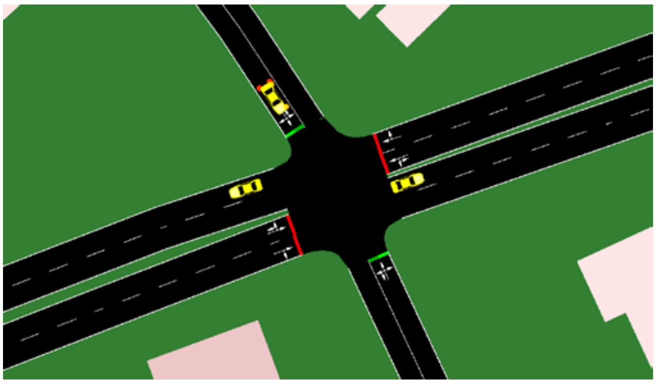

Fig. 6. Crossroad in some city, Russia (SUMO)

Let's introduce the concept of the probability of finding a road lane in a particular state (by the number of vehicles in each lane). The state observed at time t can be denoted as x ($x \in X$). In addition, we introduce the time interval τ, during which it is possible to change the state x. In this case, the value of the current time is described as $t=h\tau$, where h is the number of the step of transition between states (the process of transition between states becomes quasi-continuous with an infinite small-time interval τ), $h = 0, 1, 2, 3, \ldots, N$. Let's assume that the current state x at step h after the transition at step $h + 1$ can increase by a certain amount ε (the value of the incoming flow of machines) or decrease by an amount ξ (the value of the outgoing flow of machines) and, accordingly, be equal to x+ε or x–ξ. Then, the probabilities of finding our node in the desired state are as follows:

$P(x–\varepsilon,h)$ – the probability that the system is in a state of $(x–\varepsilon)$ on a step h;

$P(x,h)$ – the probability that the system is in a state of x;

$P(x+\xi,h)$ – the probability that the system is in a state of $(x+\xi)$.

Within the framework of this model, after each step, the state x can change by a magnitude ε or ξ. Then, the probability $P(x,h+1)$ – that in the next $(h+1)$ step the system (or process) will be in state x will be equal to (see Fig. 7):

$$P(x,h+1)=P(x–\varepsilon,h)+P(x+\xi,h)–P(x,h) \tag{5}$$

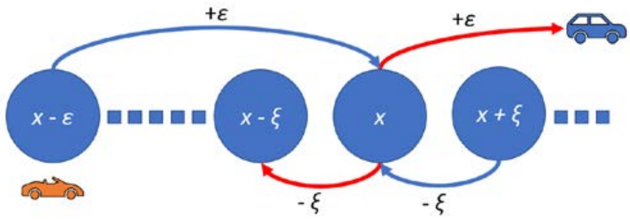

Fig. 7. Diagram of possible transitions between system (or process) states at $h+1$ step

Let us explain equation (1) and the diagram shown in Fig. 2. The probability of transition to state x at step $h+1$ is $P(x, h + 1)$, and is determined by the sum of the probabilities of transitions to this state from states $(x–\varepsilon)$ and $(x+\xi)$, i.e. $P(x - \varepsilon, h) + P(x + \xi, h)$ in which the system was located in the previous step h, but minus the probability of transition $P(x, h)$ of the system from state x (in which it was in the previous step h) to any other state at the next step $h + 1$.

Given that $t=h\cdot\tau$, where t is the process time, h is the step number, τ is the duration of one step, we will go from h to t. Expand equation (1) into a Taylor series near the point x.

Further, moving from probability-to-probability density $(\rho(x,t) = \frac{dP(x,t)}{dx})$ and considering no more than the second derivatives $\rho(x,t) = \frac{dP(x,t)}{dx}$ of x and time t, we get (6):

$$\frac{d\rho(x,t)}{dt} = a\frac{d^2\rho(x,t)}{dx^2} - b\frac{d\rho(x,t)}{dx} - c\frac{d^2\rho(x,t)}{dt^2}, \tag{6}$$

where $a = \frac{\varepsilon^2+\xi^2}{2\tau}$, $b = \frac{\varepsilon-\xi}{\tau}$ and $c=\tau$

The term of the equation ∂P(x,t)∂t – determines the total change in state over time.

∂P(x,t)∂x an ordered transition either to a state when it increases ($\varepsilon > \xi$) or when it decreases ($\varepsilon < \xi$).

$\frac{d^2\rho(x,t)}{dx^2}$ – describes a random change in state.

$\frac{d^2\rho(x,t)}{dt^2}$ – allows you to describe the situation when there is an acceleration of state changes. In the simulated conditions, the acceleration of the propagation of congestion in transport networks is not assumed, and we will not take this term of the equation into account in the presented model.

It should be noted that the basics of our approach to modeling the stochastic dynamics of processes in complex systems were described by us in our previous works [22,23].

Formulation and solution of the boundary value problem for describing the stochastic dynamics of blocking the transport network.

Let us formulate and solve a boundary value problem to describe the operation of network nodes without considering the acceleration of state change (without a term $\frac{d^2\rho(x,t)}{dt^2}$ in equation (6)). Let's assume that when the number of machines at the node reaches a given critical value L (L is a capacitive parameter of the length of the road lane divided by the average size of the car on the lane), it goes into a blocked state and becomes available only when the size of the queue decreases to a certain value. Since we are trying to avoid this state, the first boundary condition can be chosen in the form: $\rho(x,t)_{x=L}=0$ (the condition of reflection from the boundary, the probability of detecting such a state will be different from zero, and the probability density $\rho(x,t)$ must be taken equal to zero).

Given that the number of machines on the node cannot go into the area of negative values, the second boundary condition will be of the form: $\rho(x,t)_{x=0}=0$ (here the condition of reflection from the boundary must also be met).

Since at the time of the observation started, or $t = 0$, there may already be a certain number of x_0 machines on the node, the initial condition will be set in the form:

$$\rho(x,t=0)=\delta(x-x_0)$$

Since the initial condition contains a delta function, the solution for $\rho(x,t)$ is divided into two regions: at $x > x_0$ and at $x \leq x_0$.

Using the methods of operational calculus for the probability density $\rho_1(x,t)$ and $\rho_2(x,t)$ of detecting the state of the system (node of the road network graph) in one of the values in the segment from 0 to L, we can obtain the following system of equations:

for the part, when $x \leq x_0$

$$\rho_1(x,t)=\frac{2}{L}e^{-\frac{(\varepsilon-\xi)^2 t}{4\tau(\varepsilon^2+\xi^2)}}e^{\frac{(\varepsilon-\xi)(x_0-x)}{(\varepsilon^2+\xi^2)}}\sum_{n=1}^{M}\frac{\sin\left(\pi n\frac{x}{L}\right)\sin\left(\pi n\frac{L-x_0}{L}\right)}{\cos(\pi n)}e^{-\frac{\pi^2 n^2(\varepsilon^2+\xi^2)t}{2\tau L^2}} \tag{7a}$$

for the part, when $x > x_0$

$$\rho_2(x,t)=\frac{2}{L}e^{-\frac{(\varepsilon-\xi)^2 t}{4\tau(\varepsilon^2+\xi^2)}}e^{\frac{(\varepsilon-\xi)(x-x_0)}{(\varepsilon^2+\xi^2)}}\sum_{n=1}^{M}\frac{\sin\left(\pi n\frac{x_0}{L}\right)\sin\left(\pi n\frac{L-x}{L}\right)}{\cos(\pi n)}e^{-\frac{\pi^2 n^2(\varepsilon^2+\xi^2))t}{2\tau L^2}} \tag{7b}$$

If we calculate the integral $P(L,t)$:

$$P(L,t)=\int_0^{x_0}\rho_1(x,t)dx+\int_{x_0}^{L}\rho_2(x,t)dx \tag{8}$$

then, the function $P(L,t)$ will set the probability that the state of the node of the road network at time t will be in the segment from 0 to L, i.e. blocking state will not occur (state L will not be reached).

VII. Practical solutions

To obtain the maximum time intervals at which the transport node remains operational, we use models developed considering the percolation properties of the network: models of stochastic dynamics of blocking transport network nodes. To solve the problems of improving the reliability of the city's transport network, it is proposed to use the following approach:

1. Load the road network graph from an external source, such as OSM (OpenStreetMap), using ready-made SUMO tools (Simulation of urban mobility).
2. After step 1 we transform the road network into a representation in the form of a graph. We form the nodes of the graph (lines of the road strip), we form the connections between the nodes (intersections, rebuilding).
3. For each node of the road network, we calculate a certain threshold value L of the permissible number of cars on it, through the length of a fragment of the road network divided by the average length of one car (let's take 6.5 meters as such a distance) on the road strip.
4. Let's take $\tau = 1$ as the measurement interval and calculate the value of the number of outgoing machines ξ at one step and the number of incoming machines ε at one step.
5. For each node of the constructed road network, we determine at each step the value x - the length of the queue.
6. On next step we can set the value of the permissible probability $\psi(L,t)$ of blocking the node as a result of overload (availability of a single element). As a value of $\psi(L,t)$, you can take the value of the percolation threshold, calculated by the equation $\ln\{\psi max(Qi)\} = -1.71g + 0.04$ (see Fig 5), where g value – is the inversion of the network density.
7. Using the value $\psi(L,t)$ determined from the density of the network under consideration, we equate it to $P(L,t)$, and, solving the integral equation (8), we calculate for each node the time to achieve the probability of its blocking. In this case, we use the values ξ, ε, L and x_i given (founded) for the node. The probability determines the occurrence of blockage, therefore, the Zvalue of the percolation threshold for the resulting network can be used as the value of such a probability (the percolation threshold is the probability of blocking $P(L,t)$ a *single* node at which the entire network loses conductivity).
8. Having done the procedures described in steps 4-7, we get, for each stage of modeling, a table of the predicted times for each of the network nodes to reach the permissible threshold value. Based on the capacity and size of the queues of cars, it is possible to calculate the time of passage of cars through each node (road lane).

Thus, the calculated values make it possible to predict the time of failure (overload) for each road lane in the road network, and, for those lanes that are connected to traffic light regulation, make request for modified traffic light control signals to change the number of cars entering and outgoing to the lane, and, as a result, to increase the reliability of both a separate section of the road network and the entire road network as a whole.

The application of the model of stochastic dynamics of blocking transport network nodes to obtain the predicted time of blocking a road network node with a given probability of such blocking.

In Fig. 8. there is shown a comparison of the probabilities of blocking a single node of the road network with the same model parameters (time interval τ, incoming flow ε, outgoing flow ξ), but for different capacities (lengths) of the road strip. Values with $L = 75$, $L = 80$ parking spaces get the maximum probability of going beyond the limits of the capacity L faster than more capacious ones at this parameter ($L = 90$, $L = 100$). Which is a consistent result.

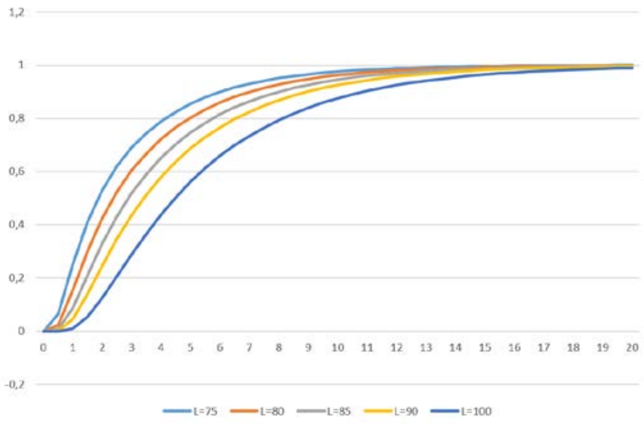

Fig. 8. Distribution of the probability of blocking a single node of the road network with the same flows (input and output), but different capacities of the road strip

The maximum operating times obtained in this way (for a given reliability of the operation of a single node of the transport network) make it possible, based on these data, to optimize (dynamically adjust) the switching intervals of traffic lights at adjacent intersections.

FINDINGS

The study shows that for different transport networks it is possible to generalize and simplify the calculation of the parameters of the availability indicators of the entire network, or its part with a high degree of correlation. Further, the obtained values of the probability of percolation transition can be applied in a model of stochastic dynamics of lane blocking and calculate with its help the values of the time to reach the blocked state. The obtained times can be used in traffic light switching control algorithms.

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